Available online at www.sciencedirect.com
SCIENCE DIRECT®
Chaos, Solitons and Fractals 23 (2005) 981–988
CHAOS SOLITONS & FRACTALS
www.elsevier.com/locate/chaos

Localization of periodic orbits of polynomial systems by ellipsoidal estimates

Konstantin E. Starkov [a,*,1], Alexander P. Krishchenko [b,2]

[a] CITEDI-IPN, Avenue del Parque 1310, Mesa de Otay, Tijuana, BC, Mexico
[b] Bauman Moscow State Technical University, 2nd Baumanskaya Street, 5, Moscow 105005, Russia

Accepted 8 June 2004

Abstract

In this paper we study the localization problem of periodic orbits of multidimensional continuous-time systems in the global setting. Our results are based on the solution of the conditional extremum problem and using sign-definite quadratic and quartic forms. As examples, the Rikitake system and the Lamb's equations for a three-mode operating cavity in a laser are considered.

1. Introduction

The localization problem of periodic orbits of nonlinear multidimensional continuous-time systems has been studied by many researchers during last years, see papers with analytical methods based on second order extremum conditions [1,4,6–8], with algebraic methods based on using algebraic dependent polynomials [12], with analytical methods based on high-order extremum conditions [13,14], see also [10] and others. Now it is well known that possessing of periodic orbits is one of essential features specifying the global dynamics of chaotical systems especially in domains containing attractors. For example, the description of a chaotic attractor with help of infinitely many unstable periodic orbits embedded in it is helpful in studies concerning the Lorenz system, see [3] and references therein.

The main contribution of this paper consists in obtaining ellipsoidal estimates for domains containing all periodic orbits and for domains having no common points with any of periodic orbits. Our methods are based on the solution of the conditional extremum problem introduced in [7] and using sign-definite quadratic and quartic forms. As examples, we consider the Rikitake system and the Lamb's equations for a three-mode operating cavity in a laser. This paper is the reworked and enlarged version of the short conference paper [15].

* Corresponding author. Tel.: +1 52 66231344; fax: +1 52 66231388.
E-mail addresses: konst@citedi.mx, konstarkov@hotmail.com (K.E. Starkov), apkri@999.ru (A.P. Krishchenko).
[1] Address: CITEDI-IPN, 2498 Roll Drive #757, San Diego, 92154 CA, USA.
[2] Tel.: +7 095 2636750; fax: +7 095 2679893.

0960-0779/$ - see front matter © 2004 Published by Elsevier Ltd.
doi:10.1016/j.chaos.2004.06.002

2. Some preliminaries and notations

We consider a polynomial system

$$\dot{x} = F(x) = b + Ax + f(x), \tag{1}$$

where f is a polynomial vector field of degree d without constant and linear terms; b is a constant vector; A is a constant $(n \times n)$-matrix; $x \in \mathbf{R}^n$ is the state vector. If f is a homogeneous quadratic vector field then (1) is called a general quadratic system. By $\varphi(x,t)$ we denote a solution of a general differentiable right side system

$$\dot{x} = v(x) \tag{2}$$

with $\varphi(x,0) = x$. For any set B in $\mathbf{R}^n$ we denote by $C\{B\}$ its complement. For any symmetric matrix T we denote by $\lambda_{\max}(T)$ its maximal eigenvalue. By diag(λ_1, λ_2, λ_3) we denote the diagonal (3×3)-matrix with elements λ_1, λ_2, λ_3 on its principal diagonal.

Let

$$f(x) = (x^{\mathrm{T}} S_1 x, \ldots, x^{\mathrm{T}} S_n x)^{\mathrm{T}}, \tag{3}$$

where matrices S_j, $j = 1,\ldots,n$, are symmetric.

Let $h(x)$ be a differentiable function such that h is not the first integral of (1) or (2). The function h will be used in the solution of the localization problem of periodic orbits and will be called localizing. By $h|_B$ we denote the restriction of h on a set $B \subset \mathbf{R}^n$. By $\Xi_1(v,h)$ we denote the set $\{x \in \mathbf{R}^n \mid L_v h(x) = 0\}$.

Let W be a subset in $\mathbf{R}^n$. Let us define $h_{\inf}(W) := \inf\{h(x) \mid x \in W\}$; $h_{\sup}(W) := \sup\{h(x) \mid x \in W\}$. We shall write $h_{\inf}$ and $h_{\sup}$ in case of $W = \mathbf{R}^n$.

In [7,8] it was proposed to apply numbers $h_{\inf}(\Xi_1(v,h))$; $h_{\sup}(\Xi_1(v,h))$ for studying a location of periodic orbits of the system (2). Namely, we have

Proposition 1 ([7,8]). *Each periodic orbit Γ of (2) is contained in the set*

$$K_h = \{h_{\inf}(\Xi_1(v,h)) \leqslant h(x) \leqslant h_{\sup}(\Xi_1(v,h))\}. \tag{4}$$

As it was made in [7,8], $h_{\inf}(\Xi_1(v,h))$ and $h_{\sup}(\Xi_1(v,h))$ were computed by using the Lagrange multiplier method in case of the Lorenz system. We construct the Lagrange function $\mathscr{L} = h - \mu L_v h$. Then corresponding critical points are found from the system

$$\frac{\partial \mathscr{L}}{\partial x_s} = 0; \quad s = 1, \ldots, n; \quad L_v h = 0. \tag{5}$$

By X we denote the set of $x_* \in \mathbf{R}^n$ for which (x_*, μ) is a solution of (5) for some μ. Therefore

$$h_{\inf}(\Xi_1(v,h)) = \inf_{x \in X} h(x); \quad h_{\sup}(\Xi_1(v,h)) = \sup_{x \in X} h(x). \tag{6}$$

3. Bounds $h_{\inf}(\Xi_1(v,h))$ and $h_{\sup}(\Xi_1(v,h))$: main results

Here our goal is to sharpen Proposition 1. Let $\mathscr{A}$ be a surface in $\mathbf{R}^n$. Suppose that $\mathscr{A}$ has no common points with any of periodic orbits of the system (2). This assumption will be referred to as Assumption 1.

Let $M = \Xi_1 \cap C\{\mathscr{A}\}$. We state

Proposition 2. *Let $M \neq \emptyset$. Then each periodic orbit Γ of (2) is contained in the set*

$$N_h = \{h_{\inf}(M) \leqslant h(x) \leqslant h_{\sup}(M)\}. \tag{7}$$

If $M = \emptyset$ then the system (2) has no periodic orbits.

The proof is made like in [7, Theorem 4].

Let us consider Assumption 1. Suppose that each component $\mathscr{A}_s$ of the set $\mathscr{A}$ is satisfied one of the following conditions: (1) $\mathscr{A}_s$ is an invariant surface for the phase flow of the system (2) and the restriction of the system (2) on $\mathscr{A}_s$ has no periodic orbits; (2) the set $\mathscr{A}_s$ is contained in the equilibria set of the system (2).

Clearly, in this case Assumption 1 holds. Here we remark respecting to the case (2) that a few conditions of the non-existence of periodic orbits for multidimensional systems are proposed in [14].

Example 3. Consider the system

$$\dot{x}_1 = x_1(1 - x_1 - \varepsilon\sin(x_2)),$$
$$\dot{x}_2 = f_2(x_1, x_2),$$

ε is a parameter. Let us take the localizing function $h(x) = x_1$. Let $\mathscr{A}$ be a straight line $x_1 = 0$. Then $M \neq \emptyset$. We conclude that Assumption 1 holds because $\mathscr{A}$ is invariant and has no common points with periodic orbits. Further, we obtain that Ξ_1 is described by equations $1 - h - \varepsilon\sin(x_2) = 0$ and $x_1 = 0$. Now if $0 < \varepsilon < 1$ then

$$0 = h_{\inf}(\Xi_1(v,h)) < h_{\inf}(M) = 1 - \varepsilon < h_{\sup}(\Xi_1(v,h)) = h_{\sup}(M) = 1 + \varepsilon.$$

So each periodic orbit is contained in the set $1 - \varepsilon < x_1 < 1 + \varepsilon$. Similarly, one can get a multidimensional version of this example.

We remind that a surface $g = 0$ is called semipermeable if $L_v g|_{g=0} > 0$ or $L_v g|_{g=0} < 0$. Suppose now that $\mathscr{A} = h^{-1}(0)$ is a semipermeable connected surface by the phase flow of the system (2).

Lemma 4. *If $\Xi_1(v,h)$ is a connected set then $0 = h_{\inf}(\mathscr{A}) \leqslant h_{\inf}(\Xi_1(v,h))$ or $h_{\sup}(\Xi_1(v,h)) \leqslant h_{\sup}(\mathscr{A}) = 0$.*

Proof. Indeed, since $\mathscr{A} \cap \Xi_1(v,h) = \emptyset$ we obtain that $\Xi_1(v,h) \subset h^{-1}((0,\infty))$ or $\Xi_1(v,h) \subset h^{-1}((-\infty,0))$. The latter entails the desirable conclusion. □

Lemma 5. *Let $h_{inf}(\Xi_1(v,h)) = h_{sup}(\Xi_1(v,h)) = c$. Then every periodic orbit is contained in the set $\Xi_1(v,h)$.*

Proof. Each periodic orbit has common points with the set $\Xi_1(v,h)$. Assume that Γ is a periodic orbit and $x \in \Gamma$. Then $\min_t h(\varphi(x,t)) = \max_t h(\varphi(x,t)) = c$. Therefore all points of Γ are critical points of the function $g(t) = h(\varphi(x,t))$. Hence, they are contained in the set $\Xi_1(v,h)$. □

If the system (2) has at least one equilibrium then

$$(L_v h)_{\inf} \leqslant 0 \leqslant (L_v h)_{\sup}.$$

If the system (2) has no equilibria points then we establish the following:

Proposition 6. *If $0 < (L_v h)_{\inf} := \inf\{L_v h(x) \mid x \in \Xi_1(v, L_v h)\}$ or $0 > (L_v h)_{\sup} := \sup\{L_v h(x) \mid x \in \Xi_1(v, L_v h)\}$ then the system (2) has no periodic orbits.*

Proof. Suppose that the system (2) has periodic orbits. Applying Proposition 1 to the function $g = L_v h$, we get that all periodic orbits are contained in the set $g > 0$ or $g < 0$. Further, applying Proposition 1 to the function h, we get that each periodic orbit has common points with the set $\Xi_1(v,h)$, see in [7]. This is a contradiction. □

Coming back to systems (2) possibly containing equilibria points, we can state the following property of periodic orbits.

Proposition 7. *If $(L_v h)_{inf} = 0$ or $(L_v h)_{sup} = 0$ then all periodic orbits are contained in the set*

$$\cap_{k=1}^{\infty}\{L_v^k h(x) = 0\}. \tag{8}$$

Proof. Consider the case $(L_v h)_{\inf} = 0$. Let us take any periodic orbit Γ. By [7, Theorem 1], there are points $x \in \Gamma$ such that $L_v h(x) = 0$. For each of these points the condition $(L_v h)_{\inf} = 0$ implies that $L_v^2 h(x) \geqslant 0$. By [7, Theorem 2], $L_v^2 h(x) = 0$. Now Theorem 4 in [7] entails the fact that $\Gamma \subset \Xi_1(v, L_v h)$. The latter means (8) because of local analyticity of a solution. The case $(L_v h)_{\sup} = 0$ is treated similarly. □

In what follows, we consider only systems (1). Now our goal is to discuss situations when

$$h_{\inf} > \inf_{\mathbf{R}^n} h(x)$$

or (and)

$$h_{\sup} < \sup_{\mathbf{R}^n} h(x).$$

In this case the localization (4) provides us nontrivial bounds for the localization of all periodic orbits. If each of these estimates is valid then we get the localization of all periodic orbits in the layer K_h. If both of boundary sets $h(x) = h_{\inf}$; $h(x) = h_{\sup}$ are ellipsoids then we get the ellipsoidal localization K_h.

Assume that the set $\Xi_1(F,h)$ is compact, with h be a quadratic form. In addition, let Ell be an ellipsoid containing $\Xi_1(F,h)$. Then

$$h_{\inf} > \inf_{Ell} h(x); \quad h_{\sup} < \sup_{Ell} h(x).$$

This idea will be exploited in the example of the Lamb's equations for a three-mode operating cavity in a laser.

By $\mathscr{H}$ we denote the class of functions of the type

$$h(x) = e + Cx + x^{\mathrm{T}}Px, \tag{9}$$

where $e \in \mathbf{R}; C = (c_1, \ldots, c_n)$; $h_{[2]}(x) := x^{\mathrm{T}}Px$ is the nontrivial first integral of the homogeneous quadratic system $\dot{x} = f(x)$. Then we notice that the degree of the polynomial $L_F h$ is less or equal 2 if and only if h is contained in $\mathscr{H}$, i.e.

$$x^{\mathrm{T}}Pf(x) \equiv 0. \tag{10}$$

Indeed, it follows from the formula

$$L_F h(x) = 2x^{\mathrm{T}}Pf(x) + x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + Cf(x) + CAx + +2b^{\mathrm{T}}Px + Cb. \tag{11}$$

Finding of matrices P satisfying the condition (10) is reduced to a solution of a system of linear homogeneous equations respecting to coefficients of P.

Example 8. Consider the system (1) with a quadratic f and $b = 0$. Suppose that $h(x) = x^{\mathrm{T}}Px \in \mathscr{H}$. Let the matrix $A^{\mathrm{T}}P + PA$ be sign-definite. Then it follows from (11) that

$$h|_{\Xi_1(F,h)} = h(0).$$

Therefore all periodic orbits are contained in the set $h^{-1}(0)$. If P is sign-definite then our system has no periodic orbits.

4. Compactness of the surface $L_F h(x) = 0$

In this section we examine the problem of checking that the surface

$$g(x) := L_F h(x) = 0 \tag{12}$$

is a compact set. Let $g(x) = \sum_{s=0}^{p} g_{[s]}(x)$,where $g_{[s]}$ is a sth homogeneous part of the polynomial g, $s \geqslant 1$; $g_{[0]}(x)$ is a real number.

By using [7], we remark that (12) defines a compact set if

$$\min_{\|x\|=1} g_{[p]}(x) > 0. \tag{13}$$

Let us consider two cases.

1. Let $d = 2$ in (1). Assume that h taken from (9) is contained in $\mathscr{H}$. Then the surface

$$L_F h(x) = x^{\mathrm{T}}\left(A^{\mathrm{T}}P + PA + \sum_{i=1}^{n} c_i S_i\right)x + CAx + 2b^{\mathrm{T}}Px + Cb = 0 \tag{14}$$

is compact if and only if the matrix $A^{\mathrm{T}}P + PA + \sum_{i=1}^{n} c_i S_i$ is sign-definite. Here we meet the following problem: when can we find some symmetric matrix P such that the corresponding h is contained in $\mathscr{H}$ and the matrix

$$M = A^{\mathrm{T}}P + PA + \sum_{i=1}^{n} c_i S_i$$

is sign-definite?

Suppose that one can find a couple of matrices S_{l_1}; S_{l_2} (we can consider that $l_1 = 1; l_2 = 2$ after a proper renumeration) satisfying the Stewart condition [16]: $z^*(S_1 + iS_2)z \neq 0$ for any nonzero $z \in \mathbf{C}^n$; here $i^2 = 1$. In this case there is θ such that $(\cos\theta)S_1 + (\sin\theta)S_2$ is a positive definite matrix where θ is obtained by the algorithm from [2]. Let us put $a_{1*} = -\cos\theta$; $a_{2*} = -\sin\theta$; $a_{j*} = 0$, j=3,...,n. Then $\sum a_{j*}S_j < 0$ and there is $\mu > 0$ such that

$$A^{\mathrm{T}}P + PA + \mu \sum a_{j*}S_j < 0. \tag{15}$$

Indeed, let us solve the inequality

$$\lambda_{\max}\left(A^{\mathrm{T}}P + PA + \mu \sum a_{j*}S_j\right) < 0$$

with respect to μ. By the classical Weil theorem [5],

$$\lambda_{\max}\left(A^{\mathrm{T}}P + PA + \mu \sum a_{j*}S_j\right) \leqslant \lambda_{\max}(A^{\mathrm{T}}P + PA) + \mu\lambda_{\max}\left(\sum a_{j*}S_j\right).$$

Hence, if

$$\mu > -\lambda_{\max}(A^{\mathrm{T}}P + PA)\lambda_{\max}^{-1}\left(\sum a_{j*}S_j\right),$$

then (15) is valid.

So if $c_j = \mu a_{j*}, j = 1,\ldots,n$, then the corresponding matrix $M < 0$. As a result, if the Stewart condition is valid for some pair of matrices from $\{S_s, s = 1,\ldots,n\}$ then for any function $h_{[2]} \in \mathscr{H}$ one can take C such that the surface $L_Fh = 0$ is compact which leads to a nontrivial localization given in Proposition 1.

If there is $h \in \mathscr{H}$ with $P > 0$ then this nontrivial localization is ellipsoidal, i.e. the set K_h is located between two ellipsoids defined by $h(x) = h_{\inf}$ and by $h(x) = h_{\sup}$.

The geometrical consequence of this fact is the dissipativity property of the system (1): each trajectory falls into the ellipsoid $h(x) = h_{\sup}$ and does not leave it.

Suppose now that there is $P = P^{\mathrm{T}} > 0$ such that (10) holds. Ellipsoids $h(x) = \mathrm{const}$ have the common center in the point $x_* = -0.5P^{-1}C^{\mathrm{T}}$. Without lost of generality one can suppose that e is chosen in a such way that $h(x_*) = 0$, i.e. $e = CP^{-1}C^{\mathrm{T}}/4$. If x_* is not contained in the surface $L_Fh(x) = 0$ then $h_{\inf} > 0$. Therefore all periodic orbits have no common points with the interior of the nontrivial ellipsoid $h(x) = h_{\inf}$. By computations we establish that the condition $L_Fh(x_*) \neq 0$ has the form

$$\frac{1}{4}C\left(P^{-1}A^{\mathrm{T}} + AP^{-1} + \sum_{i=1}^{n} c_iP^{-1}S_iP^{-1}\right)C^{\mathrm{T}} - \frac{1}{2}CAP^{-1}C^{\mathrm{T}} \neq 0.$$

2. Let us take a cubic 3-dimensional system $\dot{x} = F(x) = b + Ax + f_{[2]}(x) + f_{[3]}(x)$, where $f_{[i]}(x)$ is the homogeneous polynomial vector field of the degree $i = 2;$ 3. Let $f_{[3]}(x) = \sum_{i=1}^{3} x_i(x^{\mathrm{T}}S_{i1}x, x^{\mathrm{T}}S_{i2}x, x^{\mathrm{T}}S_{i3}x)^{\mathrm{T}}$, $S_{ij}^{\mathrm{T}} = S_{ij}$, $h(x) = x^{\mathrm{T}}Px$, $P^{\mathrm{T}} = P$.

By [7, Assertion 3], if

$$(L_Fh)_{[4]}(x) > 0, \quad ||x|| = 1 \tag{16}$$

or

$$(L_Fh)_{[4]}(x) < 0, \quad ||x|| = 1, \tag{17}$$

holds then the set described by $L_Fh(x) = 0$ is compact. Sufficient conditions for the existence of P with this property can be derived as follows. Let $S_{ij} = (s_{ijkm})$. We get that

$$(L_Fh)_{[4]}(x) = 2\sum_i x_ix^{\mathrm{T}}P(x^{\mathrm{T}}S_{i1}x, x^{\mathrm{T}}S_{i2}x, x^{\mathrm{T}}S_{i3}x)^{\mathrm{T}} = 2\sum_i x_i \sum_{j,k} x_jp_{jk}(x^{\mathrm{T}}S_{ik}x) = 2\sum_{i,j,k,l,m} p_{jk}s_{iklm}x_ix_jx_lx_m.$$

Suppose that coefficients of quartic monomials

$$x_i^3x_j, \quad i \neq j; \;\; x_1^2x_2x_3; \;\; x_1x_2^2x_3; \;\; x_1x_2x_3^2$$

are eliminated for some choice of the matrix P. It is expressed in terms of the overdetermined linear system of equations respecting p_{jk} with coefficients depending on s_{iklm}. Then there is a symmetric matrix $G = (g_{ij})_{i,j=1,2,3}$ such that

$$(L_Fh)_{[4]}(x) = y^{\mathrm{T}}Gy|_{y_i = x_i^2,\ i=1,2,3}.$$

Now if G is a sign-definite matrix or $g_{ii} > 0$, $g_{ij} \geqslant 0$ ($g_{ii} < 0$, $g_{ij} \leqslant 0$) then one of inequalities (16) and (17) holds. Hence, in these cases the surface $L_Fh(x) = 0$ is a compact set.

We provide computations in the following example. Suppose that $S_i := S_{ii}$ is a diagonal matrix with elements s_{ij}, $j = 1,2,3$ on its main diagonal. Besides, let $S_{ij} = 0$ for $i \neq j$; $i, j, = 1,2,3$. We take $h(x) = x^{\mathrm{T}}x$. By S we denote the matrix $(s_{ij})_{i,j=1,2,3}$. In this case

$$(L_F h)_{[4]}(x) = 2\left(\sum_{i=1}^{3} s_{jj}x_j^4 + \sum_{i\neq j}(s_{ij}+s_{ji})x_i^2x_j^2\right) = y^{\mathrm{T}}(S+S^{\mathrm{T}})y|_{y_i=x_i^2,\ i=1,2,3}.$$

So if the matrix $S + S^{\mathrm{T}}$ is sign-definite or $s_{ii}>0$, $s_{ij} \geqslant 0$ ($s_{ii} < 0$, $s_{ij} \leqslant 0$) then (12) defines a compact set.

5. Application 1: the Rikitake equations

Consider the Rikitake system [11]

$$\dot{x}_1 = -\mu x_1 + x_2x_3,$$
$$\dot{x}_2 = -\alpha x_1 - \mu x_2 + x_1x_3,$$
$$\dot{x}_3 = 1 - x_1x_2,$$

where α and μ are positive parameters.

Firstly, we mention that Hardy and Steeb [4] obtained the following result for the Rikitake system: an ellipsoidal domain of a general location containing all periodic orbits is not existed. Below in this section we find ellipsoidal domains which have no common points with any periodic orbit of the Rikitake system.

By solving (10), we get $P = \mathrm{diag}(p_{11},p_{22},p_{11} + p_{22})$ and the set $\mathscr{H}$ consists of polynomials

$$h(x) = p_{11}x_1^2 + p_{22}x_2^2 + (p_{11}+p_{22})x_3^2 + Cx + e.$$

Let us compute the function

$$L_Fh(x) = x^{\mathrm{T}}Mx + (-c_1\mu - c_2\alpha)x_1 - c_2\mu x_2 + 2(p_{11}+p_{22})x_3 + c_3,$$

where

$$M = \begin{pmatrix} -2p_{11}\mu & d/2 & c_2/2 \\ d/2 & -2p_{22}\mu & c_1/2 \\ c_2/2 & c_1/2 & 0 \end{pmatrix}$$

and $d = -2p_{22}\alpha - c_3$.

Let us specify coefficients of L_Fh:

$$c_2 = 0; \quad c_3 = -2p_{22}\alpha; \ \ p_{11} > 0; \ \ p_{22} > 0.$$

Then $P > 0$. As a result,

$$h(x) = p_{11}x_1^2 + p_{22}x_2^2 + (p_{11}+p_{22})x_3^2 + c_1x_1 - 2p_{22}\alpha x_3 + e$$
$$= p_{11}(x_1 + c_1/p_{11})^2 + p_{22}x_2^2 + (p_{11}+p_{22})(x_3 - p_{22}\alpha(p_{11}+p_{22})^{-1})^2$$

with $e = c_1^2/p_{11} + p_{22}^2\alpha^2(p_{11}+p_{22})^{-1}$. The surface $L_Fh(x) = 0$ is the set given by the equation

$$2\mu(p_{11}x_1^2 + p_{22}x_2^2) - c_1x_2x_3 + c_1\mu x_1 - 2(p_{11}+p_{22})x_3 + 2p_{22}\alpha = 0. \tag{18}$$

Surfaces $h(x) = \mathrm{const}$ are ellipsoids with centers in the point

$$x_* = (-c_1/p_{11}, 0, p_{22}\alpha(p_{11}+p_{22})^{-1}).$$

The point x_* is not contained in the surface (18) because $L_Fh(x_*) = -\mu c_1^2/p_{11} \neq 0$. For any $x \neq x_*$ we obtain that $h(x) > 0$. Hence $h_{\inf} > 0$ and all periodic orbits have no common points with the interior of the nontrivial ellipsoid $h(x) = h_{\inf}$.

6. Application 2: the Lamb's equations for a three-mode operating cavity in a laser

The Lamb's equations for a three-mode operating cavity in a laser have the form of (1), [9],

$$\dot{x}_1 = \lambda_1x_1 - a_{11}x_1^3 - a_{12}x_1x_2^2 - a_{13}x_1x_3^2,$$
$$\dot{x}_2 = \lambda_2x_2 - a_{21}x_2x_1^2 - a_{22}x_2^3 - a_{23}x_2x_3^2,$$
$$\dot{x}_3 = \lambda_3x_3 - a_{31}x_3x_1^2 - a_{32}x_3x_2^2 - a_{33}x_3^3.$$

Suppose that all parameters λ_s; a_{ij} are positive numbers. We take the localizing function $h(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$. By computations, we obtain that

$$L_F h = \sum_{s=1}^{3} \lambda_s x_s^2 - \sum_{s=1}^{3} a_{ss} x_s^4 - (a_{12} + a_{21}) x_1^2 x_2^2 - (a_{13} + a_{31}) x_1^2 x_3^2 - (a_{23} + a_{32}) x_2^2 x_3^2.$$

It is clear that $h_{\inf} = 0$. Further, $L_F h$ can be estimated in the following way:

$$L_F h \leqslant \sum_{s=1}^{3} \lambda_s x_s^2 - \sum_{s=1}^{3} a_{ss} x_s^4 = \sum_{s=1}^{3} (-a_{ss}(x_s^2 - \lambda_s a_{ss}^{-1}/2)^2 + \lambda_s^2 a_{ss}^{-1}/4).$$

Now if $L_F h(x) = 0$ then for any j, $j = 1,2,3$, we obtain that $a_{jj}(x_j^2 - \lambda_j a_{jj}^{-1}/2)^2 \leqslant \sum_{s=1}^{3} \lambda_s^2 a_{ss}^{-1}/4$. So

$$x_j^2 \leqslant \frac{1}{2} \sqrt{\sum_{s=1}^{3} \lambda_s^2 a_{ss}^{-1}/a_{jj}} + \lambda_j a_{jj}^{-1}/2, \quad j = 1, 2, 3.$$

The latter estimate entails

$$h_{\sup} \leqslant \frac{1}{2} \sum_{j=1}^{3} \left(\sqrt{\sum_{s=1}^{3} \lambda_s^2 a_{ss}^{-1}/a_{jj}} + \lambda_j a_{jj}^{-1}/2 \right).$$

Thus, each periodic orbit is contained in the sphere of the radius

$$\sqrt{\sum_{j=1}^{3} \left(\sqrt{\sum_{s=1}^{3} \lambda_s^2 a_{ss}^{-1}/a_{jj}} + \lambda_j a_{jj}^{-1}/2 \right)}$$

centered at the origin.

7. Conclusions

In this paper the localization problem of periodic orbits of nonlinear multidimensional continuous-time systems is examined. We present new results concerning ellipsoidal estimates for domains containing all periodic orbits and for domains without common points with any of periodic orbits. Previous results concerning ellipsoidal localization contained in [15] have been further improved in this paper in the sense that better bounds for ellipsoidal estimates are obtained here. The key idea of our approach consists in solving the conditional extremum problem described in [7] and using sign-definite quadratic and quartic forms. Our results are applied to the analysis of the Rikitake system and the Lamb's equations for a three-mode operating cavity in a laser. Future researching challenge is to get ellipsoidal estimates for periodic orbits of polynomial systems by using polynomials of even degrees $2d \geqslant 4$. Also, it is of interest to apply this approach to the localization of compact invariant domains (attractors or repellers).

Acknowledgment

The second author was supported by the grant 02–01–00704 from the Russian Foundation for Basic Research.

References

[1] Babloyantz A, Krishchenko A, Nosov A. Analysis and stabilization of nonlinear systems. Comput Math Appl 1997;34:356–69.
[2] Crawford CR, Moon YS. Finding a positive definite linear combination of two Hermitian matrices. Linear Algebra Appl 1983;51:37–48.
[3] Franceschini V, Giberti C, Zheng Z. Characterization of the Lorenz attractor by unstable periodic orbits. Nonlinearity 1993:251–8.
[4] Hardy Y, Steeb WH. The Rikitake two-disk dynamo system and domains with periodic orbits. Internat. J Theor Phys 1999;38:2413–7.
[5] Horn RA, Johnson CR. Matrix analysis. Cambridge: Cambridge University Press; 1986.

[6] Kocarev L, Parlitz U, Hu B. Lie derivatives and dynamical systems. Chaos, Solitons & Fractals 1998;9:1359–66.
[7] Krishchenko AP. Localization of limit cycles. Differen Equat 1995;31:1826–33.
[8] Krishchenko AP. Estimation of domains with cycles. Comput Math Appl 1997;34:325–32.
[9] Lamb Jr WE. Theory of an optical maser. Phys Rev 1964;134:A1429–50.
[10] Pogromsky A, Nijmeijer H, Rooda K. A negative criterion for a class of hybrid systems, In Proc. 16th Int. Symp. Mathematical Theory of Networks and Systems MTNS'2004, Leuven, Belgium, 2004.
[11] Rikitake T. Oscillations of a system of disk dynamos. Proc Cambridge Philos Soc 1958;54:89–105.
[12] Starkov K. Some results on periodic orbits of polynomial systems. Stab Control: Theory Appl 2002;4:30–42.
[13] Starkov K. Localization/nonuniqueness condition of periodic orbits of polynomial systems and its applications. In: CD-ROM with Proc. Int. Conf. Physics and Control, St. Petersburg, Russia. 2003.
[14] Starkov K. Localization of periodic orbits of autonomous systems based on high order extremum conditions. Math Probl Eng, in press.
[15] Starkov K, Krishchenko AP. Ellipsoidal estimates for domains containing all periodic orbits of general quadratic systems, In: Proc. 16th Int. Symp. Mathematical Theory of Networks and Systems MTNS'2004, Leuven, Belgium, 2004.
[16] Stewart GW. Perturbation bounds for definite generalized eigenvalue problem. Linear Algebra Appl 1979;23:69–85.