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LOCALIZATION OF COMPACT INVARIANT SETS OF NONLINEAR SYSTEMS WITH APPLICATIONS TO THE LANFORD SYSTEM

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In this paper, we examine the localization problem of compact invariant sets of systems with the differentiable right-side. The localization procedure consists in applying the iterative algorithm based on the first order extremum condition originally proposed by one of authors for periodic orbits. Analysis of a location of compact invariant sets of the Lanford system is realized for all values of its bifurcational parameter.

Keywords: Localization; compact set; invariant set; nonlinear system; Lanford system.

1. Introduction

The study of compact invariant sets is one of the important topics in the qualitative theory of ordinary differential equations closely related to analysis of a long-time behavior of a system, see e.g. [Foias *et al.*, 1996] and references therein. During the past years, the focus of interest for many researchers has been towards finding some geometrical bounds for attractors, periodical orbits and chaotic dynamics of a nonlinear autonomous differentiable right-side system

$$\dot{x} = f(x), \quad x = (x_1, \ldots, x_n)^T \in \mathbf{R}^n, \tag{1}$$

$f(x) = (f_1(x), \ldots, f_n(x))^T \in C^\infty(\mathbf{R}^n)$. Mainly, in the existing literature this problem has been solved with the help of Lyapunov-type functions. One can mention a paper [McMillen, 1998] concerning the Rikitake system and papers [Doering & Gibbon, 1995; Leonov *et al.*, 1996; Li *et al.*, 2005, Pogromsky *et al.*, 2003; Swinnerton-Dyer, 2001] concerning the Lorenz system.

The method for finding families of semipermeable surfaces was proposed in [Giacomini & Neukirch, 1997] and then developed in [Neukirch & Giacomini, 2000] and in other publications of these authors. In this paper, we consider the application of the localization method of periodic orbits [Krishchenko, 1995, 1997a, 1997b, 1997c; Krishchenko & Shalneva, 1998, 2000; Starkov & Krishchenko, 2004, 2005] to studies of a location of compact invariant sets of the Lanford system. Our basic tool is to apply the iterative method of the localization of periodic orbits originally introduced in [Krishchenko, 1997a] to the localization problem of compact invariant sets of the system (1).

Here, when we talk about a localization we have in mind the following problem: find the set $\Omega \subset \mathbf{R}^n$ (a localization set) that contains all compact invariant sets of the system (1).

The structure of the paper is as follows. In Sec. 2, we formulate basic definitions and present main assertions applied in the localization process. Other sections discuss the Lanford system. As it was noted in [Hassard *et al.*, 1981], this system had been constructed by W. F. Lanford in a private communication in connection with the analysis of an infinite system of ordinary differential equations proposed in [Hopf, 1948] for a study of a fluid dynamic turbulence model. Section 3 contains main results on a localization of compact invariant sets of the Lanford system. We describe in Sec. 4 the way to improve a localization set in the case $1/2 < v < 1$ for which the most diverse dynamics is observed. In Sec. 5, we compare our results with those of [Hassard *et al.*, 1981] and [Nikolov & Bozhkov, 2004] respecting periodic orbits and the attractor of the Lanford system. Also, we provide additional information concerning a localization of compact invariant sets of the Lanford system. In Sec. 6 we present conclusions.

2. Some Preliminaries

We start from two basic concepts, see e.g. in [Guckenheimer & Holmes, 1983], of qualitative theory of ordinary differential equations used in this paper.

By $\varphi(x,t)$ we denote a solution of (1), with $\varphi(x,0) = x$ for any $x \in \mathbf{R}^n$.

Definition 1. A set $G \subset \mathbf{R}^n$ is called invariant for (1) if for any $x \in G$ we have: $\varphi(x,t) \in G$ for all $t \in \mathbf{R}$.

Definition 2. The union of equlibrium points with trajectories connecting them is referred to as heteroclinic orbits when they connect disctinct points and homoclinic orbits when they connect a point to itself.

Compact invariant sets can contain equlibrium points, periodic orbits, heteroclinic orbits, homoclinic orbits and trajectories of more complex structure. We define a maximal (with respect to inclusions) compact invariant set of (1) as a compact invariant set containing any compact invariant set of (1). A maximal compact invariant set may not exist.

In this section we describe localization sets which contain all compact invariant sets of the system (1). The localization of invariant subsets such as periodic orbits, homoclinic orbits, heteroclinic orbits, invariant tori inside an invariant set claims to apply additional ideas which is beyond the scope of this paper.

Let $f = \sum_{i=1}^{n} f_i(x)\partial/\partial x_i$ be a vector field on $\mathbf{R}^n$ corresponding to the system (1). Let $L_f h(x) = \sum_{i=1}^{n} f_i(x)\partial h(x)/\partial x_i$ be a Lie derivative of the function $h \in C^\infty(\mathbf{R}^n)$ with respect to the vector field f. We define a set

$$S_h = \{x : L_f h(x) = 0\}.$$

Below we shall use notations:

$$\begin{aligned} h_{\sup} &= \sup_{S_h} h(x), \\ h_{\inf} &= \inf_{S_h} h(x). \end{aligned} \tag{2}$$

For any function $h \in C^\infty(\mathbf{R}^n)$ the following assertions are valid.

Proposition 1. *Let Q be a set in $\mathbf{R}^n$. If $S_h \cap Q = \emptyset$ then the system (1) has no compact invariant sets (totally) contained in Q.*

Proposition 2. *Any compact invariant set G of the system (1) is contained in the set*

$$\Omega_h = \{x : h_{\inf} \le h(x) \le h_{\sup}\}.$$

If the set Ω_h is compact then the system (1) has a maximal compact invariant set which is contained in Ω_h.

Corollary 3. *Any compact invariant set of the system (1) is contained in the set $\Omega = \{\cap\Omega_h, h \in C^\infty(\mathbf{R}^n)\}$. If the set Ω is compact then the system (1) has a maximal compact invariant set which is contained in Ω.*

Sometimes one can find the compact localizing set containing all periodic orbits using only one function, see papers of Krishchenko and Starkov mentioned above. This requirement can be satisfied, but it is difficult to find a corresponding function. The next methodology based on using two and more functions is more promising.

Theorem 4. *Let $h_m(x), m = 0, 1, 2, \ldots$ be a sequence of functions from $C^\infty(\mathbf{R}^n)$. Sets*

$$\Omega_0 = \Omega_{h_0}, \quad \Omega_m = \Omega_{m-1} \cap \Omega_{m-1,m}, \quad m > 0,$$

with

$$\begin{aligned} \Omega_{m-1,m} &= \{x : h_{m,\inf} \le h_m(x) \le h_{m,\sup}\}, \\ h_{m,\sup} &= \sup_{S_{h_m} \cap \Omega_{m-1}} h_m(x), \\ h_{m,\inf} &= \inf_{S_{h_m} \cap \Omega_{m-1}} h_m(x), \end{aligned}$$

contain any compact invariant set of the system (1) and

$$\Omega_0 \supseteq \Omega_1 \supseteq \cdots \supseteq \Omega_m \supseteq \ldots.$$

Proofs of these results are realized in the same way like in [Krishchenko, 1997a, 1997b].

3. Localization of Compact Invariant Sets of the Lanford System

Let us consider the Lanford system

$$\begin{aligned}\dot{x}_1 &= (v-1)x_1 - x_2 + x_1x_3,\\ \dot{x}_2 &= x_1 + (v-1)x_2 + x_2x_3,\\ \dot{x}_3 &= vx_3 - x_1^2 - x_2^2 - x_3^2,\end{aligned} \tag{3}$$

where v is a parameter. Let f be the vector field of the Lanford system.

A. Case $v = 0$.

For the function $h_0(x) = x_3$ we find

$$L_f h_0(x) = -x_1^2 - x_2^2 - x_3^2.$$

Therefore the set

$$S_{h_0} = \{x : x_1^2 + x_2^2 + x_3^2 = 0\} = \{(0,0,0)^T\}$$

coincides with the equilibrium point of the system, and in this case the maximal compact invariant set of the Lanford system consists of the unique equilibrium point $x = 0$.

Our localization analysis presented below is based on Theorem 4.

B. Case $v < 0$.

We use functions $h_0(x) = x_3$ and $h_1(x) = (x_1^2 + x_2^2)/2$.

(0.) If $v < 0$ then we have for the function $h_0(x) = x_3$ that

$$L_f h_0(x) = vx_3 - x_1^2 - x_2^2 - x_3^2,$$

$$S_{h_0} = \left\{x : x_1^2 + x_2^2 + \left(x_3 - \frac{v}{2}\right)^2 = \frac{v^2}{4}\right\},$$

$$h_{0,\sup} = \sup_{S_{h_0}} h_0(x) = 0, \quad h_{0,\inf} = \inf_{S_{h_0}} h_0(x) = v,$$

and all compact invariant sets are located in the set

$$\Omega_0 = \Omega_{h_0} = \{x : v \le x_3 \le 0\}.$$

(1.) Consider the second function $h_1(x) = (x_1^2 + x_2^2)/2$. Then

$$\begin{aligned} L_f h_1(x) &= (v - 1 + x_3)(x_1^2 + x_2^2),\\ S_{h_1} &= \{x : x_1^2 + x_2^2 = 0\} \cup \{x : x_3 = 1 - v\}\\ &= \{x : x = (0,0,x_3)\} \cup \{x : x_3 = 1 - v\},\\ S_{h_1} \cap \Omega_0 &= \{x = (0,0,x_3) : v \le x_3 \le 0\end{aligned}$$

since $v < 0$. We find

$$h_{1,\sup} = \sup_{S_{h_1} \cap \Omega_0} h_1(x) = 0,$$

$$h_{1,\inf} = \inf_{S_{h_1} \cap \Omega_0} h_1(x) = 0,$$

$$\Omega_{0,1} = \{x : x_1^2 + x_2^2 = 0\} = \{x : x_1 = 0, x_2 = 0\},$$

$$\Omega_1 = \Omega_0 \cap \Omega_{0,1} = \{x : x_1 = 0, x_2 = 0, v \le x_3 \le 0\}.$$

All compact invariant sets belong to the compact set Ω_1. The Lanford system has the form

$$\dot{x}_1 = 0, \dot{x}_2 = 0, \dot{x}_3 = vx_3 - x_3^2$$

on the set Ω_1. Therefore the set Ω_1 is an invariant one. The set Ω_1 is the heteroclinic orbit of the system connecting two equilibrium points $(0,0,0)^T$ and $(0,0,v)^T$. The set Ω_1 is the maximal compact invariant set of Lanford system in the case $v < 0$.

C. Case $v > 0$.

(0.) We choose $h_0(x) = x_3$. Then

$$L_f h_0(x) = vx_3 - x_1^2 - x_2^2 - x_3^2,$$

$$S_{h_0} = \left\{x : x_1^2 + x_2^2 + \left(x_3 - \frac{v}{2}\right)^2 = \frac{v^2}{4}\right\}$$

$$h_{0,\sup} = \sup_{S_{h_0}} h_0(x) = v, \quad h_{0,\inf} = \inf_{S_{h_0}} h_0(x) = 0$$

and all compact invariant sets are located in the set

$$\Omega_{h_0} = \{x : 0 \le x_3 \le v\}.$$

Let us consider points of planes $x_3 = 0$ and $x_3 = v$ different from the equilibrium points $(0,0,0)^T$ and $(0,0,v)^T$. For these points we obtain from the third equation of the Lanford system that

$$\dot{x}_3 = -x_1^2 - x_2^2 < 0.$$

It means that trajectories of the Lanford system intersect transversally the planes $x_3 = 0$ and $x_3 = v$ outside the equilibrium points $(0,0,0)^T$ and $(0,0,v)^T$. Therefore we have that all compact invariant sets are located in the set

$$\Omega_0 := \{x : 0 < x_3 < v\} \cup (0,0,0)^T \cup (0,0,v)^T. \tag{4}$$

(1.) Now we choose $h_1(x) = (x_1^2 + x_2^2)/2$. Then

$$\begin{aligned} L_f h_1(x) &= (v - 1 + x_3)(x_1^2 + x_2^2),\\ S_{h_1} &= \{x : x_1^2 + x_2^2 = 0\} \cup \{x : x_3 = 1 - v\}\\ &= \{x : x = (0,0,x_3)\} \cup \{x : x_3 = 1 - v\}, \qquad (5)\\ S_{h_1} \cap \Omega_0 &= \{x = (0,0,x_3) : 0 \le x_3 \le v\}\\ &\quad \cup \{x : x_3 = 1 - v, 0 < x_3 < v\}.\end{aligned}$$

The condition $0 < 1 - v < v$ is equivalent to $0.5 < v < 1$.

(1.1.) Consider the case $\{0 < v \le 0.5\} \cup \{v \ge 1\}$. The set

$$\{x : x_3 = 1 - v, 0 < x_3 < v\}$$

is empty. Hence we find

$$S_{h_1} \cap \Omega_0 = \{x = (0,0,x_3) : 0 \le x_3 \le v\},$$
$$h_{1,\sup} = \sup_{S_{h_1} \cap \Omega_0} h_1(x) = 0,$$
$$h_{1,\inf} = \inf_{S_{h_1} \cap \Omega_0} h_1(x) = 0,$$
$$\Omega_{0,1} = \{x : x_1^2 + x_2^2 = 0\} = \{x : x_1 = 0, x_2 = 0\},$$
$$\Omega_1 = \Omega_0 \cap \Omega_{0,1} = \{x = (0,0,x_3) : 0 \le x_3 \le v\}.$$

As in the case $v < 0$, the obtained set Ω_1 (a heteroclinic orbit) is the maximal compact invariant set of the Lanford system in the case $\{0 < v \le 0.5\} \cup \{v \ge 1\}$.

(1.2.) Consider the case $0.5 < v < 1$. In this case the set

$$\{x : x_3 = 1 - v, 0 < x_3 < v\}$$

is a plane and we obtain

$$h_{1,\sup} = \sup_{S_{h_1} \cap \Omega_0} h_1(x) = +\infty,$$
$$h_{1,\inf} = \inf_{S_{h_1} \cap \Omega_0} h_1(x) = 0,$$
$$\Omega_{0,1} = \mathbf{R}^3,$$
$$\Omega_1 = \Omega_{0,1} \cap \Omega_0 = \Omega_0.$$

(2.) Let $h_2(x) = (1/2)(x_1^2 + x_2^2 + x_3^2)$, $0.5 < v < 1$. We get

$$L_f h_2 = (x_1^2 + x_2^2 + x_3^2)(v - 1) + x_3^2 - x_3^3$$

and

$$S_{h_2} = \{x : (x_1^2 + x_2^2 + x_3^2)(v - 1) = -x_3^2 + x_3^3\},$$
$$S_{h_2} \cap \Omega_1 = \{x : (x_1^2 + x_2^2 + x_3^2)(v - 1) = -x_3^2 + x_3^3, 0 \le x_3 \le v\}.$$

In order to find $h_{2,\inf}$, $h_{2,\sup}$ we consider

$$\frac{x_3^2}{2} \le h_2(x)|_{S_{h_2} \cap \Omega_1} = \frac{1}{2(v-1)}(x_3^3 - x_3^2),$$

where $0 \le x_3 \le v$.

Under $0 \le x_3 \le v$ the inequality

$$\frac{x_3^2}{2} \le \frac{1}{2(v-1)}(x_3^3 - x_3^2)$$

is fulfilled. Finding inf and sup of the function

$$\frac{1}{2(v-1)}(x_3^3 - x_3^2),$$

under the condition $x_3 \in [0, v]$, we get that

$$h_{2,\inf} = \inf_{S_{h_2} \cap \Omega_1} h_2(x) = 0,$$

$$h_{2,\sup} = \sup_{S_{h_2} \cap \Omega_1} h_2(x) = \begin{cases} \dfrac{v^2}{2}, & 0.5 < v \le \dfrac{2}{3}, \\[2mm] \dfrac{2}{27(1-v)}, & \dfrac{2}{3} < v < 1, \end{cases}$$

$$\Omega_{1,2} = \{x : x_1^2 + x_2^2 + x_3^2 \le 2h_{2,\sup}\}.$$

So all compact invariant sets are located in the set

$$\Omega_2 = \Omega_1 \cap \Omega_{1,2} = \{x : x_1^2 + x_2^2 + x_3^2 \le 2h_{2,\sup}, 0 < x_3 < v\} \cup (0,0,0)^T \cup (0,0,v)^T. \quad (6)$$

(3.) Consider the case $0.5 < v < 1$. Let

$$h_3(x) = h_1(x) = \frac{x_1^2 + x_2^2}{2}.$$

Then $S_{h_3} = S_{h_1}$, see (5), and we have for Ω_2 from (6):

$$\begin{aligned} S_{h_3} \cap \Omega_2 = \{&x = (0,0,x_3) : x_1^2 + x_2^2 + x_3^2 \le 2h_{2,\sup}, 0 < x_3 < v\} \cup (0,0,0)^T \\ &\cup (0,0,v)^T \cup \{x : x_1^2 + x_2^2 + x_3^2 \le 2h_{2,\sup}, 0 < x_3 < v, x_3 = 1 - v\} \\ = \{&x = (0,0,x_3) : x_3^2 \le 2h_{2,\sup}, \times 0 \le x_3 \le v\} \\ &\cup \{x : x_1^2 + x_2^2 \le 2h_{2,\sup} - (1-v)^2, x_3 = 1 - v\} \\ = \{&x = (0,0,x_3) : 0 \le x_3 \le v\} \cup \{x : x_1^2 + x_2^2 \le 2h_{2,\sup} - (1-v)^2, x_3 = 1 - v\}. \end{aligned}$$

In order to prove the last equality of the sets we consider two cases.

Case 1. Let $v \in (0.5, 2/3]$. Then $h_{2,\sup} = v^2/2$, and the inequality $x_3^2 \le 2h_{2,\sup} = v^2$ follows from $0 \le x_3 \le v$. In addition,

$$2h_{2,\sup} - (1-v)^2 = 2v - 1 \ge 0.$$

Hence

$$S_{h_3} \cap \Omega_2 = \{x = (0,0,x_3) : 0 \le x_3 \le v\} \cup \{x : x_1^2 + x_2^2 \le 2v - 1, x_3 = 1 - v\}.$$

Case 2. Let $v \in [2/3, 1)$. Then $h_{2,\sup} = 2/(27(1 - v))$, and we have the inequality $2h_{2,\sup} \ge v^2$ for these values of v. Therefore the inequality $x_3^2 \le 2h_{2,\sup}$ follows from $0 \le x_3 \le v$. In addition,

$$2h_{2,\sup} - (1-v)^2 = \frac{4}{27(1-v)} - (1-v)^2 > 0.$$

Hence

$$S_{h_3} \cap \Omega_2 = \{x = (0,0,x_3) : 0 \le x_3 \le v\} \cup \left\{ x : x_1^2 + x_2^2 \le \frac{4}{27(1-v)} - (1-v)^2, x_3 = 1-v \right\}.$$

In these two cases we get

$$h_{3,\sup} = \sup_{S_{h_3}\cap\Omega_2} h_3(x) = h_{2,\sup} - \frac{(1-v)^2}{2},$$

$$h_{3,\inf} = \inf_{S_{h_3}\cap\Omega_2} h_3(x) = 0,$$

$$\Omega_{2,3} = \{x : x_1^2 + x_2^2 \le 2h_{2,\sup} - (1-v)^2\}.$$

If $0.5 < v < 1$ all compact invariant sets are located in the set

$$\Omega_3 = \Omega_2 \cap \Omega_{2,3} = \{x : x_1^2 + x_2^2 + x_3^2 \le 2h_{2,\sup}, 0 < x_3 < v, x_1^2 + x_2^2 \le 2h_{2,\sup} - (1-v)^2\} \cup (0,0,0)^T \cup (0,0,v)^T. \tag{7}$$

In order to simplify this form of the localization set we consider the same two cases.

Case 1. Let $v \in (0.5, 2/3]$. Then $2h_{2,\sup} = v^2$. Hence, the localization set is given by

$$\{x : x_1^2 + x_2^2 + x_3^2 \le v^2, 0 < x_3 \le v, x_1^2 + x_2^2 \le 2v - 1\} \cup (0,0,0)^T. \tag{8}$$

Case 2. Let $v \in (2/3, 1)$. Then $2h_{2,\sup} = 4/(27(1-v)) > v^2$ and for these values of v we have that the localization set is given by

$$\left\{ x : x_1^2 + x_2^2 + x_3^2 \le \frac{4}{27(1-v)}, 0 < x_3 < v, x_1^2 + x_2^2 \le \frac{4}{27(1-v)} - (1-v)^2 \right\} \cup (0,0,0)^T \cup (0,0,v)^T. \tag{9}$$

4. A Closer Look at the Localization Set in Case $0.5 < v < 1$

We have found exactly the maximal compact invariant sets of the Lanford system in the case $\{v \le 0.5\} \cup \{v \ge 1\}$ and the localization set (7) for $0.5 < v < 1$. The case $0.5 < v < 1$ is the most interesting since the Lanford system exhibits a chaotical behavior in some neighborhood of $v = 2/3$, see [Nikolov & Bozhkov, 2004].

As it follows from (7)–(9), the localizing set Ω_3 is monotonically extended up to the set $\{x : 0 < x_3 < v\} \cup (0,0,0)^T \cup (0,0,v)^T$, with $v \to 1-0$. Below we demonstrate that it is possible to improve the localization of all compact invariant sets by using the localizing function $H(x) = 0.5(x_1^2 + x_2^2) + (v-1)x_3$.

Since the localizing set Ω_3 is computed by sufficiently long computations it is easy to see that a continuation of the iterative procedure with $h_4 := H(x)$ leads to cumbersome computations. Instead of this, let us consider another localizing procedure with the function $h_0(x)$ used above and a new function $h_1(x)$. Below in our computations we use the set Ω_0 defined in the formula (4). Let us take the function $\tilde{h}_1(x) = H(x)$ as the new function $h_1(x)$. It leads to the following computations at the first step of the new localizing procedure.

1. Let $\tilde{h}_1(x) = 0.5(x_1^2 + x_2^2) + (v-1)x_3$, $0.5 < v < 1$. We obtain

$$L_f\tilde{h}_1 = x_3(x_1^2 + x_2^2 + x_3(1-v) + v(v-1))$$

and

$$S_{\tilde{h}_1} = \{x : x_3 = 0\} \cup \{x : x_1^2 + x_2^2 + x_3(1-v) + v(v-1) = 0\},$$

$$S_{\tilde{h}_1} \cap \Omega_0 = \{x : x_1^2 + x_2^2 + x_3(1-v) + v(v-1) = 0, 0 < x_3 \le v\} \cup (0,0,0)^T.$$

Let $P = \{x : x_1^2 + x_2^2 + x_3(1-v) + v(v-1) = 0, 0 < x_3 \le v\}$. In order to find $\tilde{h}_{1,\inf}$, $\tilde{h}_{1,\sup}$ we consider

$$\tilde{h}_1(x)|_P = \frac{3}{2}(v-1)x_3 + \frac{v(1-v)}{2},$$

where $0 < x_3 \le v$, and $\tilde{h}_1(0,0,0) = 0$. Therefore

$$\tilde{h}_{1,\inf} = v(v-1), \quad \tilde{h}_{1,\sup} = \frac{v(1-v)}{2},$$

$$\Omega_{0,1} = \left\{ x : -\frac{v}{2} + \frac{x_1^2 + x_2^2}{2(1-v)} \le x_3 \le v + \frac{x_1^2 + x_2^2}{2(1-v)} \right\},$$

and a new localizing set

$$\tilde{\Omega}_1 = \Omega_0 \cap \Omega_{0,1} = \left\{ x : 0 < x_3 < v, x_3 \ge -\frac{v}{2} + \frac{x_1^2 + x_2^2}{2(1-v)} \right\} \cup (0,0,0)^T \cup (0,0,v)^T.$$

If $v \to 1-0$ the set $\tilde{\Omega}_1$ containing the heteroclinic orbit $\Gamma(v) = \{x : x_1 = 0, x_2 = 0, 0 \le x_3 \le v\}$

collapses into the heteroclinic orbit $\Gamma(1) = \{x : x_1 = 0, x_2 = 0, 0 \leq x_3 \leq 1\}$ of the Lanford system for $v = 1$. This is also true for the localizing set

$$\begin{aligned}\tilde{\Omega} &= \tilde{\Omega}_1 \cap \Omega_3 \\ &= \left\{x : 0 < x_3 < v, x_3 \geq -\frac{v}{2} + \frac{x_1^2 + x_2^2}{2(1-v)},\ x_1^2 + x_2^2 \right. \\ &\quad + x_3^2 \leq 2h_{2,\sup},\ x_1^2 + x_2^2 \leq 2h_{2,\sup} \\ &\quad \left. - (1-v)^2 \right\} \cup (0,0,0)^T \cup (0,0,v)^T. \end{aligned} \tag{10}$$

The last localization can be improved for compact invariant sets having no common points with $\Gamma(v)$ by applying the third localizing procedure with the functions $h_0(x)$ and

$$\hat{h}_1(x) = \frac{x_3}{x_1^2 + x_2^2}.$$

It leads to the following computations at the first step of the third localizing procedure applied to the set $\{x_1^2 + x_2^2 \neq 0\}$.

1. We compute

$$\begin{aligned} L_f \hat{h}_1 &= \frac{vx_3 - x_1^2 - x_2^2 - x_3^2 - 2x_3(v - 1 + x_3)}{x_1^2 + x_2^2} \\ &= \hat{h}_1(-v - 3x_3) + 2\hat{h}_1 - 1. \end{aligned}$$

Therefore

$$\begin{aligned} S_{\hat{h}_1} &= \{x :\ vx_3 - x_1^2 - x_2^2 - x_3^2 \\ &\quad - 2x_3(v - 1 + x_3) = 0, x_1^2 + x_2^2 \neq 0\} \\ &= \left\{x : x_1^2 + x_2^2 + 3\left(x_3 - \frac{(2-v)}{6}\right)^2 \right. \\ &\quad \left. = \frac{(2-v)^2}{12},\ x_1^2 + x_2^2 \neq 0 \right\}, \end{aligned}$$

$$\begin{aligned} S_{\hat{h}_1} \cap \Omega_0 = S_{\hat{h}_1}, &\quad \hat{h}_1|_{S_{\hat{h}_1}} = \frac{1}{2 - v - 3x_3}, \\ \inf \hat{h}_1|_{S_{\hat{h}_1}} &= \inf_{S_{\hat{h}_1}} \frac{1}{2 - v - 3x_3} = \frac{1}{2 - v}, \\ \sup \hat{h}_1|_{S_{\hat{h}_1}} &= \sup_{S_{\hat{h}_1}} \frac{1}{2 - v - 3x_3} = +\infty, \\ \Omega_{0,1} &= \left\{x :\ x_3 \geq \frac{x_1^2 + x_2^2}{2 - v}\right\}, \end{aligned}$$

$$\begin{aligned} \hat{\Omega} = \Omega_0 \cap \Omega_{0,1} &= \left\{x :\ 0 < x_3 < v, x_3 \geq \frac{x_1^2 + x_2^2}{2 - v}\right\} \\ &\quad \cup (0,0,0)^T \cup (0,0,v)^T \end{aligned}$$

and we obtain the localizing set

$$\begin{aligned} \tilde{\Omega} \cap \hat{\Omega} = \{x :\ 0 < x_3 < v,\ x_3 \geq -\frac{v}{2} + \frac{x_1^2 + x_2^2}{2(1-v)}, \\ x_3 \geq \frac{x_1^2 + x_2^2}{2 - v}, x_1^2 + x_2^2 + x_3^2 \leq 2h_{2,\sup}, x_1^2 + x_2^2 \\ \leq 2h_{2,\sup} - (1-v)^2\} \cup (0,0,0)^T \cup (0,0,v)^T. \end{aligned} \tag{11}$$

for compact invariant sets having no common points with $\Gamma(v)$.

The localization set (11) was derived under the assumption that we localize compact invariant sets outside $\{x_1^2 + x_2^2 \neq 0\}$. Nevertheless, the set (11) contains $\Gamma(v)$ and all other compact invariant sets from $\mathbf{R}^3 \backslash (0,0,0)^T$ as well.

5. General Remarks

1. The iterative localization method of compact invariant sets (Theorem 4) works efficiently for the Lanford system because of two reasons. Firstly, by our choice of localizing functions we avoid a solution the conditional extremum problem introduced in (2) by the Lagrange multiplies method. Instead of this, we have solved the univariate extremum problem or have found a solution from geometrical considerations. Secondly, we have found a localizing function h ($h = h_1$ in notations given above) for which the polynomial $L_f h_1$ is decomposed into two factors such that their corresponding variables are independent. It leads to sufficiently easy computations of resulting localization sets with the help of Theorem 4.

2. By a numerical simulation, it was demonstrated in [Nikolov & Bozhkov, 2004] that a chaotic attractor really exists for values of the bifurcational parameter v in a small neighborhood of the point $2/3$. It corresponds to the existence of a family of bifurcating tori for $v = 2/3$ approximately computed for the Lanford system in cylindrical coordinates in [Hassard *et al.*, 1981]. The formula (10) provides a localization of this attractor.

3. It was found in [Hassard *et al.*, 1981] by using cylindrical coordinates

$$\begin{aligned} x_1 &= r\cos\theta; \\ x_2 &= r\sin\theta; \\ x_3 &= x_3 \end{aligned}$$

that the Lanford system can be written as

$$\begin{aligned}\dot{r} &= r(v - 1 + x_3);\\ \dot{x}_3 &= vx_3 - r^2 - x_3^2;\\ \theta &= t\end{aligned} \tag{12}$$

and, evidently, possesses the periodic orbit

$$\begin{aligned}x_1(t) &= \sqrt{(1-v)(2v-1)}\cos t,\\ x_2(t) &= \sqrt{(1-v)(2v-1)}\sin t,\\ x_3 &= 1 - v.\end{aligned} \tag{13}$$

This corresponds to our localization results presented in formulae (7), (10), (11) because the formula (13) defines a real-valued periodic (nonconstant) function only if $1/2 < v < 1$.

4. It follows from the formula for $S_{\hat{h}_1}$ that it is described by

$$\hat{h}_1^2(x) + \frac{v-2}{3(x_1^2 + x_2^2)}\hat{h}_1(x) + \frac{1}{3(x_1^2 + x_2^2)} = 0.$$

We note that the discriminant of this quadratic equation is non-negative iff

$$x_1^2 + x_2^2 \le \frac{(v-2)^2}{12}. \tag{14}$$

By Proposition 1, we deduce from the formula (14) that each compact invariant set lying outside $x_1 = x_2 = 0$ has points in the cylinder defined in (14). So in the case $0.5 < v < 1$ these compact invariant sets are placed inside the set (11) and have common points with the set (14).

5. We remark that all localizing functions applied above can be easily written in a rational way in cylindrical coordinates and in this case the same localization results for the Lanford system is established by using (12) as well. Nevertheless, if we apply a function

$$g(x_3, r) = \frac{x_3}{r}$$

to the system (12) we get the final improvement of a localization of compact invariant sets in some cases. Indeed, if ρ is the vector field corresponding to the two first equations in (12) then we get that

$$L_\rho g = \frac{x_3 - r^2 - 2x_3^2}{r}$$

and the set $L_\rho g = 0$ is given by $g^2 - gr^{-1}/2 + 1/2 = 0$. Its discriminant is $1 - 8r^2$. Again by applying Proposition 1, we deduce that each compact invariant set lying outside $x_1 = x_2 = 0$ has points in the cylinder

$$x_1^2 + x_2^2 \le \frac{1}{8}. \tag{15}$$

So in case $0.5 < v < 1$ all these compact invariant sets are placed inside the set (11) and have common points with the set (15).

At last, by comparing bounds (14) and (15), we obtain that

$$\frac{(v-2)^2}{12} > \frac{1}{8}$$

for $0.5 < v < 2 - \sqrt{3/2} \approx 0.775$. So in the case $v = 2/3$ the bound in the formula (15) provides us a more precise information about a location of compact invariant sets than the bound in (14). Otherwise, in the case $2 - \sqrt{3/2} \le v < 1$, the bound given in (14) is more precise than in (15).

6. By using other rational functions and Proposition 1, one can obtain additional information concerning a location of compact invariant sets. Namely, let us apply $h = x_1/x_2$. Then it is easy to obtain $L_f h = -1 - h^2$. Thus there are no compact invariant sets without common points with the plane $x_2 = 0$. Similarly, we apply $h = x_2/x_1$. Then it is easy to obtain $L_f h = 1 + h^2$. Thus there are no compact invariant sets without common points with the plane $x_1 = 0$. Hence all compact invariant sets contain common points with both planes $x_1 = 0$ and $x_2 = 0$.

6. Conclusions

In this article we have examined the localization problem of compact invariant sets of the system (1) by using the iterative localization method elaborated earlier for localizing periodic orbits. We have shown that our approach works effectively in the case of the analysis of the Lanford system for all values of the bifurcational parameter v. The most interesting results presented here concern localizing the chaotic attractor which has been recently described by a numerical procedure in the existing literature.

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