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Localization of compact invariant sets of the Lorenz system

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Abstract

The problem of finding domains in the state space of a nonlinear system which contain all compact invariant sets is considered. Such domains are computed for the Lorenz system by using different localizing functions.

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1. Introduction

The Lorenz equations is one of the most famous models of nonlinear dynamics exhibiting chaos:

$$\begin{aligned}
\dot{x}_1 &= -\sigma x_1 + \sigma x_2,\\
\dot{x}_2 &= r x_1 - x_2 - x_1 x_3,\\
\dot{x}_3 &= x_1 x_2 - b x_3,
\end{aligned}\tag{1}$$

where σ, r and b are positive parameters. These equations have been derived by Lorenz in [1]. The problem concerning bounds for the attractor of the system (1) was considered by Lorenz and thereafter has been examined in many papers which can be separated into three groups.

(1) The first one is based on using Lyapunov type functions. This approach dates back directly to [1] in which some ellipsoidal localization of the attractor was obtained; see also [2]. Later, we mention ellipsoidal bounds described in the book of Sparrow [3] by applying quadratic functions

$$\begin{aligned}
h(x) &= r x_1^2 + \sigma x_2^2 + \sigma (x_3 - 2r)^2,\\
h(x) &= r x_1^2 + \sigma x_2^2 + \sigma \bigl(x_3 - r(r-1)\bigr)^2,\\
h(x) &= x_1^2 + \sigma x_2^2 + \sigma x_3^2.
\end{aligned}\tag{2}$$

More recently, the attention of many authors was attracted to the problem of a more precise computation of ellipsoidal estimates for the Lorenz attractor. One can consult in papers by Leonov [4,5], Leonov with coauthors [6], Doering and Gibbon [7,8], Swinnerton-Dyer [9]; see also [10] and [11] in which a few additional quadratic functions h have been used. A localization of the Lorenz attractor by two ellipsoids, a pair of parabolic cylinders, an elliptic cylinder and an elliptic cone was obtained by Doering and Gibbon in [7]. Leonov with coauthors in [6] computed some set containing the Lorenz attractor in case $\sigma \geqslant \max\{1; b\}$.
Let us remind that a trapping region is a set such that each trajectory either enters this set and thereafter never leaves it or tends to its boundary. Therefore one may localize a trapping region. We mention here that a localization of the trapping region was got in [9] and in [12] with help of applying quadratic functions and a quartic. The latter results are similar to results of Giacomini and Neukirch cited below.

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(2) Giacomini and Neukirch elaborated the method of semipermeable surfaces which is based on using constructions of Darboux polynomials in a combination with a special choice of its coefficients, see in [13,14]. This approach was applied to the localization of the Lorenz attractor. Giacomini and Neukirch found a set of semipermeable hyperboloids such that the Lorenz attractor is squeezed between them. Besides, they described a few semipermeable families of ellipsoids with a generalization of results of Lorenz, Sparrow, Doering and Gibbon. In addition, they constructed other families of semipermeable surfaces of more complicated structure describing the location of the Lorenz attractor.

(3) The third approach concerns a method of Krishchenko in [15] on localizing of periodic orbits of a differentiable right side system based on applying the first order extremum conditions. Recently, some new developments of this method have been obtained, see papers of Starkov and Krishchenko [16,17]. In [18] ellipsoidal and other localization results of periodic orbits of the Lorenz system can be found. It is easy to see and actually it was made in [19] that all localization sets which are computed by this method contain not only all periodic orbits but all compact invariant sets as well. Thus estimates of [18] provides us also the localization set containing the Lorenz attractor.

The main goal of our Letter is to get a localization of all compact invariant sets of the Lorenz system for all positive values of parameters σ, r and b by using the approach described in [15–18]. Up to the authors knowledge, this problem has not been examined yet in the existing literature. For the case including the standard one ($\sigma = 10$; $r = 28$; $b = 8/3$) our localization bounds are simple, tight and can be easily computed.

The structure of this Letter is as follows. Some preliminaries and useful results are presented in the next section. Section 3 contains results on localizing compact invariant sets of the Lorenz system. Section 4 contains a discussion of results obtained in this Letter and relations between different localization methods.

2. Some preliminaries and notations

We consider a smooth system

$$\dot{x} = F(x), \tag{3}$$

where $x \in \mathbf{R}^n$, $F(x) = (F_1(x), \dots, F_n(x))^T$, and $F_i(x) \in C^\infty(\mathbf{R}^n)$, $i = 1, \dots, n$.

Let $h(x) \in C^\infty(\mathbf{R}^n)$ be a function such that h is not the first integral of the system (3). The function h is used in the solution of the localization problem of compact invariant sets and is called a localizing function. By $h|_B$ we denote the restriction of h on a set $B \subset \mathbf{R}^n$. By S_h we denote the set

$$S_h = \{x \in \mathbf{R}^n\colon\ L_F h(x) = 0\}, \tag{4}$$

where $L_F h(x)$ is a Lie derivative with respect to vector field F corresponding to the system (3). In [15,18] it was proposed to apply numbers

$$h_{\inf}(S_h) := \inf_{x \in S_h} h(x), \qquad h_{\sup}(S_h) := \sup_{x \in S_h} h(x)$$

for studying a location of periodic orbits of the system (3). Namely, we have

Theorem 1 ([15,18]). *For any $h(x) \in C^\infty(\mathbf{R}^n)$ each periodic orbit of system (3) contains at least two points of the set S_h.*

Theorem 2 ([15,18]). *For any $h(x) \in C^\infty(\mathbf{R}^n)$ each periodic orbit of system (3) is contained in the set*

$$\Omega_h = \{x \in \mathbf{R}^n\colon\ h_{\inf}(S_h) \leqslant h(x) \leqslant h_{\sup}(S_h)\}. \tag{5}$$

Any of sets Ω_h will be called a localizing set. Below we shall briefly write: $h_{\inf} := h_{\inf}(S_h)$; $h_{\sup} := h_{\sup}(S_h)$.

It is easy to see and in fact it was noted in [19] that Theorems 1, 2 remain valid if we formulate them respecting compact invariant sets instead of periodic orbits.

The following generalization of Theorem 2 will be helpful for a localization of all compact invariant sets.

Theorem 3. *Let O be an open set in $\mathbf{R}^n$ and let us take a set $Q \subset O$. For any $\phi(x) \in C^\infty(O)$ each compact invariant set of the system (3) contained in Q is contained in the set*

$$\Omega_\phi(Q) = \{x\colon\ \phi_{\inf}(S_\phi \cap Q) \leqslant \phi(x) \leqslant \phi_{\sup}(S_\phi \cap Q)\}, \tag{6}$$

as well where

$$\phi_{\inf}(S_\phi \cap Q) = \inf_{S_\phi \cap Q} \phi(x), \qquad \phi_{\sup}(S_\phi \cap Q) = \sup_{S_\phi \cap Q} \phi(x).$$

In our analysis of the Lorenz system the following lemma is helpful as well.

Lemma 4. *If all compact invariant sets are contained in the set K and in the set L then they are contained in $K \cap L$ as well.*

3. Localization of compact invariant sets of the Lorenz system

3.1. Ellipsoidal localization

Let F be the vector field of the Lorenz system.

Proposition 5 ([18]). *For a polynomial of degree 2 of the form*

$$h(x) = \alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 + 2\delta x_1 x_2 + 2\varepsilon x_1 x_3 + 2\mu x_2 x_3 + 2\theta x_1 + 2\nu x_2 + 2\lambda x_3 + \tau,$$

following three conditions

(1) $\deg L_F h(x) = 2$;

(2) *terms of second degree of function $L_F h(x)$ define a negative definite quadratic form*;

(3) *terms of second degree of function $h(x)$ define a positive definite quadratic form are fulfilled iff*

$$\alpha > 0, \qquad \beta = \gamma > 0, \qquad \delta = \varepsilon = \mu = 0,$$

and

$$|\alpha\sigma + \beta r + \lambda| < \sqrt{4\alpha\beta\sigma - \nu^2/b}, \tag{7}$$

i.e.,

$$h(x) = \alpha x_1^2 + \beta x_2^2 + \beta x_3^2 + 2\theta x_1 + 2\nu x_2 + 2\lambda x_3 + \tau, \tag{8}$$

where $\alpha > 0$, $\beta > 0$, and inequality (7) holds. If the conditions (1)–(3) hold then the set Ω_h is bounded by an ellipsoid.

For the function (8)

$$L_F h = -2\alpha\sigma x_1^2 - 2\beta x_2^2 - 2\beta b x_3^2 + 2(\alpha\sigma + \beta r + \lambda)x_1 x_2 - 2\nu x_1 x_3 + 2(\nu r - \theta\sigma)x_1 - 2(\nu - \theta\sigma)x_2 - 2\lambda b x_3 \tag{9}$$

and the surface S_h is an ellipsoid

$$\alpha\sigma x_1^2 + \beta x_2^2 + \beta b x_3^2 - (\alpha\sigma + \beta r + \lambda)x_1 x_2 + \nu x_1 x_3 - (\nu r - \theta\sigma)x_1 + (\nu - \theta\sigma)x_2 + \lambda b x_3 = 0. \tag{10}$$

The condition $h_{\inf} \leqslant h(x)$ is fulfilled for all x and does not impose any restrictions. On the other hand, we obtain $h_{\sup} < +\infty$, and the condition $h(x) \leqslant h_{\sup}$ defines the set Ω_h limited by the ellipsoid

$$\alpha(x_1 + \theta/\alpha)^2 + \beta(x_2 + \nu/\beta)^2 + \beta(x_3 + \lambda/\beta)^2 = h_{\sup} - \tau + \theta^2/\alpha + \nu^2/\beta + \lambda^2/\beta, \tag{11}$$

with the center $(-\theta/\alpha, -\nu/\beta, -\lambda/\beta)$.

All compact invariant sets of the system (1) place inside the ellipsoid (11). All trajectories from the external set relatively ellipsoid (11) fall into the ellipsoid (11) and remain there. The function (8) decreases on trajectories of the system (1) outside of the ellipsoid (10), and it grows inside the ellipsoid (10). The chaotic motion of the system (1) is possible only inside the ellipsoid (11).

There is a long list of quadratic localizing functions giving ellipsoidal localization sets including (2) and others considered in [4,6,7,9,11,18].

Below we choose h in a more specific form

$$h(x) = x_1^2 + \beta x_2^2 + \beta(x_3 + \lambda/\beta)^2, \tag{12}$$

where $\beta > 0$; β and λ satisfy conditions (7) under $\alpha = 1$, $\nu = 0$. To find $h_{\inf}$ and $h_{\sup}$ we take the Lagrange function in the form $\mathcal{L}(x, \mu) = h(x) + \mu L_F h(x)/2$, with $L_F h(x)/2 = -\sigma x_1^2 - \beta x_2^2 - \beta b x_3^2 + (\sigma + \beta r + \lambda)x_1 x_2 - \lambda b x_3$. Applying the necessary extremum condition, we form the system of equations

$$\frac{\partial\mathcal{L}}{\partial x_1} = 2(1 - \mu\sigma)x_1 + \mu(\sigma + \beta r + \lambda)x_2 = 0,$$
$$\frac{\partial\mathcal{L}}{\partial x_2} = \mu(\sigma + \beta r + \lambda)x_1 + 2\beta(1 - \mu)x_2 = 0,$$
$$\frac{\partial\mathcal{L}}{\partial x_3} = 2\beta(1 - \mu b)x_3 + \lambda(2 - \mu b) = 0,$$
$$L_F h(x) = 0. \tag{13}$$

Let us find all real solutions of this system. Consider the linear system respecting $x_1; x_2$ consisting of the first two equations in (13). The determinant of this system Det is equal to zero provided μ satisfies the equation $\alpha_1\mu^2 + 2\alpha_2\mu + 4\beta = 0$, with $\alpha_1 = 4\beta\sigma - (\sigma + \beta r + \lambda)^2 > 0$, $\alpha_2 = -2\beta(1 + \sigma)$. This equation has two real roots $\mu_1; \mu_2$, $\mu_1 > \mu_2$, for $\beta > 0$.

Suppose that the system (13) has the solution $x_1 = x_2 = 0$. Then we obtain from the last equation of (13) that (a) $x_3 = 0$ and $\lambda(2 - \mu b) = 0$, $p_1 := h(0, 0, 0) = \lambda^2\beta^{-1}$ or (b) $x_3 = -\lambda\beta^{-1}$ and $\lambda\mu = 0$; $h(0, 0, -\lambda\beta^{-1}) = 0$.

If the solution of (13) satisfies the condition $x_1^2 + x_2^2 > 0$ then $\mathrm{Det} = 0$ and therefore $\mu = \mu_1$ or $\mu = \mu_2$. Let us demonstrate that if $\sigma > 1, b < \sigma$ then (13) has no real solutions provided $\mu = \sigma^{-1}$ or(and) $\mu = b^{-1}$.

(1) Assume that $\mu = \sigma^{-1} < 1$. Then by using the first equation of (13) we get that $\mu(\sigma + \beta r + \lambda)x_2 = 0$ and thus $\sigma + \beta r + \lambda = 0$ or $x_2 = 0$.
Now if $x_2 = 0$ then since $x_1^2 + x_2^2 > 0$ we obtain from the second equation of (13) that $\lambda = -\sigma - \beta r$. Further, we get from the third equation of (13) that

$$x_3 = -\lambda\frac{2 - \mu b}{2\beta(1 - \mu b)} = \frac{(\sigma + r\beta)(2\sigma - b)}{2\beta(\sigma - b)} > 0.$$

The last equation of (13) has no solutions because

$$\sigma x_1^2 + b x_3(\lambda + x_3\beta) = \sigma x_1^2 + b^2 x_3\frac{\sigma + r\beta}{2(\sigma - b)} > 0.$$

If $\sigma + \beta r + \lambda = 0$ then we get from the second equation of (13) that $x_2 = 0$, but this case has been already examined.

(2) Now let $\mu = b^{-1}$. Then by using the third equation of (13), we have that $\lambda(2 - \mu b) = 0$ and therefore $\lambda = 0$. Besides, $1 - \mu\sigma = (b - \sigma)b^{-1} < 0$ and $x_1 = -kx_2$, with

$$k = \frac{\mu(\sigma + \beta r + \lambda)}{2(1 - \mu\sigma)} = \frac{\sigma + r\beta}{2(\sigma - b)} < 0.$$

Eliminating x_1 from the last equation of (13) we obtain the equation

$$\left(\sigma k^2 + (\sigma + \beta r)k + \beta\right)x_2^2 + \beta b x_3^2 = 0.$$

This equation has no solutions with $x_2^2 + x_3^2 > 0$ because $\sigma k^2 + (\sigma + \beta r)k + \beta > 0$.
Thus in case $x_2^2 + x_3^2 > 0$ we get that $\mu = \mu_1$ or $\mu = \mu_2$ but $\mu \neq 1/\sigma$ and $\mu \neq 1/b$. Then we obtain from the third equation of (13) that

$$x_3 = -\lambda\frac{2 - \mu b}{2\beta(1 - \mu b)}, \tag{14}$$

and $x_1 = -kx_2$, with

$$k = \frac{\mu(\sigma + \beta r + \lambda)}{2(1 - \mu\sigma)}. \tag{15}$$

Eliminating x_1 from the last equation of (13) we obtain the equation

$$\left(\sigma k^2 + (\sigma + \beta r)k + \beta\right)x_2^2 + b x_3(\beta x_3 + \lambda) = 0. \tag{16}$$

Thus the system (13) has no more than 4 real solutions for which $x_1^2+x_2^2>0$. Now by p_2 we denote the maximal value of the function (12) computed at points of these solutions. If there are no real solutions then we put $p_2=0$.
As a result, the maximal value of the polynomial (12) is a function of parameters β; λ and has a form $s(\beta,\lambda)=\max\{p_1;\,p_2\}$. Hence, the localizing set Ω_h is contained in the ellipsoid

$$\Omega_h=\{x\colon\ x_1^2+\beta x_2^2+\beta(x_3+\lambda/\beta)^2\leqslant s(\beta,\lambda)\}. \tag{17}$$

3.2. Localization by a parabolic cylinder

Let us take $\alpha=1$, $\beta=0$, $\theta=0$, $\nu=0$, $\lambda=-\sigma$, $\tau=0$ in (8), (9). Then we have that $h=x_1^2-2\sigma x_3$, $L_F h=-2\sigma x_1^2+2b\sigma x_3$. Therefore the set S_h is given by $x_3=b^{-1}x_1^2$ and $h\,|_{S_h}=(1-2\sigma b^{-1})x_1^2$.

Thus if $b=2\sigma$ then all compact invariant sets are located in $\Omega_h=\{x\colon\ x_1^2-2\sigma x_3=0\}$. If $b>2\sigma$ then $h_{\inf}=0$, $h_{\sup}=+\infty$, and all compact invariant sets are located in $\Omega_h=\{x\colon\ x_1^2-2\sigma x_3\geqslant 0\}$. If $b<2\sigma$ then $h_{\sup}=0$, $h_{\inf}=-\infty$, and all compact invariant sets are located in

$$\Omega_h=\{x\colon\ x_1^2-2\sigma x_3\leqslant 0\}. \tag{18}$$

3.3. Localization by an elliptic cylinder

Let us take $\alpha=0$, $\beta=0.5$, $\theta=0$, $\nu=0$, $\lambda=-r/2$, $\tau=0$ in (8), (9). Then we have that $h=(x_2^2+x_3^2)/2-rx_3$, $L_F h=-x_2^2-bx_3^2+brx_3$. Therefore the set S_h is given by $x_2^2=-bx_3^2+brx_3$ and

$$h\,|_{S_h}=\frac{1}{2}(1-b)x_3^2+\frac{r}{2}(b-2)x_3,$$

where $-bx_3^2+brx_3\geqslant 0$. Thus $h_{\inf}=-r^2/2$,

$$h_{\sup}=0,\quad \text{if } 0<b\leqslant 2,$$
$$h_{\sup}=\frac{r^2(b-2)^2}{8(b-1)},\quad \text{if } b>2,$$

and all compact invariant sets are located in

$$\Omega_h=\{x\colon\ x_2^2+x_3^2-2rx_3\leqslant 2h_{\sup}\}$$

because $-r^2\leqslant x_2^2+x_3^2-2rx_3$ for all x_2, x_3.

In the case $b>2$

$$\Omega_h=\left\{x\colon\ x_2^2+(x_3-r)^2\leqslant\frac{r^2b^2}{4(b-1)}\right\},$$

and therefore for all compact invariant sets we have the localization set

$$\omega_1:=\left\{x\colon\ |x_2|\leqslant\frac{rb}{2\sqrt{b-1}},\ |x_3-r|\leqslant\frac{rb}{2\sqrt{b-1}}\right\}. \tag{19}$$

3.4. Applications of Theorem 3

Theorem 6. *If the set $\omega=\{x\colon\ |x_2|\leqslant Y\}$ is a localization set for the system (1) then each compact invariant set of the Lorenz system is contained in the set $\Omega=\{x\colon\ |x_1|\leqslant Y\}$.*

Proof. Let us apply Theorem 3 in case $Q=\omega$, $\phi(x)=h(x)=x_1$. For $h=x_1$ the set $S_h=\{x\colon\ x_1-x_2=0\}$, and $h|_{S_h}=x_2$. Then $h_{\inf}(S_h\cap\omega)=-Y$, $h_{\sup}(S_h\cap\omega)=Y$, and the set $\Omega_h=\Omega$. Therefore all compact invariant sets are located in Ω. □

Theorem 7. *In the case $2<b<2\sigma$ the set*

$$\omega_2:=\left\{x\colon\ |x_1|\leqslant\frac{rb}{2\sqrt{b-1}},\ |x_2|\leqslant\frac{rb}{2\sqrt{b-1}},\ \frac{x_1^2}{2\sigma}\leqslant x_3\leqslant r\left(1+\frac{b}{2\sqrt{b-1}}\right)\right\} \tag{20}$$

is a localization set for all invariant compact sets.

Proof. It follows from (19) that the set $\omega=\{x\colon\ |x_2|\leqslant rb/(2\sqrt{b-1})\}$ is localizing and therefore by Theorem 6 the set $\Omega=\{x\colon\ |x_2|\leqslant rb/(2\sqrt{b-1})\}$ is also localizing. The set (20) is an intersection of Ω with sets (18) and (19). □

The localizing set (20) contains all invariant compacts of the Lorenz system. If $\sigma=10$; $b=8/3$; $r=28$ then this localizing set contains the attractor and gives the estimate $|x_1|\leqslant 28.918$ respecting the variable x_1. Apart from this bound for the Lorenz attractor in [6] one can find may be the most precise known bound $|x_1|\leqslant 21.412$ and the bound $|x_1|<21$ according to results of numerical simulation. This indicates the fact that bounds (20) for all compact invariant sets may be possible to improve. In order to see this let us use Theorem 3 in case $Q=\Omega_h$ taken from (17) with $\phi=x_1$. Then we have that $\phi_{\sup}(S_\phi\cap Q)=\sup\{x\colon\ x_1\in S_\phi\cap Q\}$, where $S_\phi=\{x\colon\ x_1=x_2\}$. Therefore it follows from (17) that $\phi_{\sup}(S_\phi\cap Q)=\max x_2$, where x_2 is a root of the equation

$$(1+\beta)x_2^2+\beta\big(x_3+\lambda\beta^{-1}\big)^2=s(\beta,\lambda).$$

The latter equation entails

$$|x_2|\leqslant K(\beta,\lambda):=\sqrt{\frac{s(\beta,\lambda)}{1+\beta}}.$$

Hence $\phi_{\sup}(S_\phi\cap Q)=K(\beta,\lambda)$, $\phi_{\inf}(S_\phi\cap Q)=-K(\beta,\lambda)$ and the localizing set is described by

$$\Omega_\phi(\Omega_h)\cap\Omega_h=\{x\colon\ |x_1|\leqslant K(\beta,\lambda)\}\cap\Omega_h, \tag{21}$$

with β and λ be satisfied (7) and $\beta>0$. Now if $\sigma=10$; $b=8/3$; $r=28$ then by applying numerical methods of minimization one can obtain that

$$\inf_\lambda K(1,\lambda)=26.986. \tag{22}$$

Therefore, all compact invariant sets are located in $\{x\colon\ |x_1|\leqslant 26.986<28.918\}$.

Remark 1. Finding the bound (22) is based on a calculation of the value of the function $s(\beta,\lambda)$ with help of formulae obtained as a result of a solution of the system (13). For admissible values of β and λ, i.e. for $\beta>0$, $\lambda\in(\lambda^-,\lambda^+)$, $\lambda^\pm=-\sigma-\beta r\pm\sqrt{4\beta\sigma}$, we have by a definition that $s(\beta,\lambda)=\max\{p_1;\,p_2\}$, with $p_1 = \lambda^2\beta^{-1}$. The way of a calculation of p_2 is as follows. We find roots μ_1 and μ_2 of the equation $\mathrm{Det} = 0$. For $\mu = \mu_1$ we find values x_3 (14), k defined in (15), roots $x_{2\pm}$ of the equation (16) and $x_{1\pm} = -kx_{2\pm}$. If $x_{2\pm}$ are real then we calculate values of the function (12) at points $(x_{1\pm}, x_{2\pm}, x_3)$. Similar calculations should be realized for $\mu = \mu_2$ as well. Now p_2 is equal to maximum of calculated values of the function (12). If roots $x_{2\pm}$ of Eq. (16) are complex then we take $p_2 = 0$.

Now we give comments concerning a comparison of our bounds with some of existing ones. We note that our bounds in (20) for $|x_2|$; $|x_3|$ obtained for the localization of all compact invariant sets are the same as bounds in [6] for $|x_2|$; $|x_3|$ obtained for the localization of the Lorenz attractor only. As for our bounds for $|x_1|$, see (20) and (22), they are not as good as given in [6]. For example, if we take the Lorenz system with standard values of its parameters then the Leonov's bound 21.412 for $|x_1|$ is better than our bound 26.986. But again, our bounds are got for localizing of all compact invariant sets while the Leonov's bound is got for the localization of the Lorenz attractor only.

4. Conclusions

In this Letter we continue the elaboration of the localization method described in our previous papers and apply it for a localization of all compact invariant sets of the famous Lorenz system. Therefore our results are valid for bounding the Lorenz attractor.

However the Lorenz system can have other compact invariant sets. They can be various separatrices, periodic orbits with stable and unstable manifolds, but all of them are contained in localizing sets found in our Letter. In case of the existence of such compact invariant sets bounds derived for the attractor can be worse than bounds obtained by other methods. For example, a separatrix connecting two equilibrium points is always located in the localizing set and therefore all its points satisfy corresponding bounds. But this separatrix can cross semipermeable surfaces and thus estimates based on them are not necessarily fulfilled for points of such a separatrix. At the same time there are common features for the considered localization method and the method based on semipermeable surfaces. For example, if $c > h_{\sup}$ then each connected component of the set $\{x\colon\ h(x) = c\}$ is semipermeable because its intersection with S_h is empty and therefore the function $L_F h(x)$ preserves its sign on it, i.e., it is positive or negative.

We meet the similar situation with the Lyapunov-based approach which is based on the estimate of attractors with help of using the boundedness property of trajectories of the system. Therefore comparing our localizing method with the Lyapunov-based approach points to their distinctions and common features as well. To see this, we mention that there are no additional assumptions in Theorem 3 respecting the equation (3) or a function ϕ. This means that our method is not reduced to the Lyapunov-based approach. Indeed, suppose that at least one of bounds $h_{\inf}(S_h)$; $h_{\sup}(S_h)$ is finite in the formula (5), e.g., it is $h_{\inf}(S_h)$. Then functions h and $L_f h$ may have the same sign on the set $\{h(x) = h_{\inf}(S_h)\}$. For example, consider the localization problem for the Rössler system

$$\dot{x} = -y - z, \qquad \dot{y} = x + ay, \qquad \dot{z} = b + xz - cz,$$

with $a; c > 0$. We apply the localizing function $h = x^2 + y^2 + 2z$ [18]. Then $L_f h = 2(ay^2 + b - cz)$. By a substitution of the expression for z obtained from the equation $L_f h = 0$ into h we get that

$$h\,|_{S_h} = x^2 + 2ac^{-1}y^2 + 2bc^{-1}.$$

Then $h_{\inf}(S_h) = 2bc^{-1}$ and $\Omega_h = \{2bc^{-1} \leqslant x^2 + y^2 + 2z\}$. Now we note that $h > 0$ and $L_f h \geqslant 0$ on the paraboloid $\{2bc^{-1} = x^2 + y^2 + 2z\}$.

At the same time the Lyapunov-based approach can be based on functions $V(x)$ satisfying the differential inequality

$$L_F V(x) + \vartheta\big(V(X) - c\big) \leqslant 0,$$

with $\vartheta > 0, c \in \mathbf{R}$. It follows from this inequality that the attractor of the system is contained in the set $\{x\colon\ V(x) \leqslant c\}$. But the same inequality implies that $V_{\sup}(S_V) \leqslant c$. Hence all compact invariant sets of the system are located in the set $\Omega_V = \{x\colon\ V_{\inf}(S_V) \leqslant V(x) \leqslant V_{\sup}(S_V)\}$ which leads to the more precise bound for the attractor.

The main feature of methods under discussion consists in the fact that they do not use numerical integration of the considered system of differential equations. It is important for analysis of chaotical systems. Though these methods are directed toward a solution of different problems they can be applied for obtaining bounds of attractors. These methods are united by the fact that each of them handles with some function. If this function is chosen then obtaining corresponding results does not cause fundamental difficulties. At the same time we have to use some heuristic methods for a choice of a localizing function or parametric classes of localizing functions. Hence one of possible directions for future investigations can be related with obtaining results concerning the choice of functions used.

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