Parity and Cobordisms of Virtual Knots

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Outline

Parity and Diagram–Valued Invariants

The notion of parity, first introduced in 2009, allowed not only to strengthen invariants and create new ones, but also to create *picture–valued invariants*. That means that the invariant object is valued in graphs, not in numbers.

A perfect example of an invariant of that kind is *parity bracket* which discovery led, for example, to the proof of non-triviality of free knots. The bracket is defined in a combinatorial way using *smoothings*. The bracket invariance leads to the following equality:

$$[K] = K,$$

where K on the left means a knot, and on the right — one particular diagram of that knot.

The "cat principle"

The existence of such invariants means that in some cases it is possible to obtain some important information about a knot looking at its single diagram. For example, the following principle often holds:

The "cat principle" states that if a diagram is complicated enough, it realises itself as a subdiagram in any other diagram of the same knot.

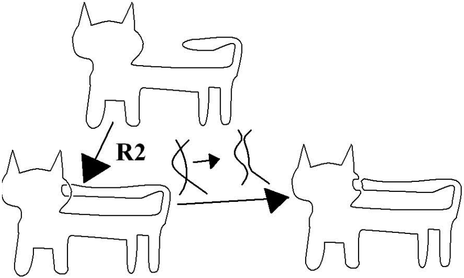

Figure: Cats

The Central Result

The main result described in this talk realises this "picture–valued" goal for another important property of knots: their sliceness. That is, looking at a diagram of a knot with certain properties (namely, an odd diagram) one can understand, whether this knot is slice or not.

Basic Definitions

Recall that two classical knots K_1, K_2 are called concordant if they can be connected with a cylinder in $\mathbb{R}^3 \times [0,1]$ so that K_1 lies in $\mathbb{R}^3 \times \{0\}$ and K_2 lies in $\mathbb{R}^3 \times \{1\}$.

This notion can be easily expanded to deal with free knots. Saying "free knot" we mean an equivalence class of decorated 4-valent graphs (with one unicursal component) modulo Reidemeister moves.

Definition

Two free knots Γ_1, Γ_2 are called cobordant if there exists a spanning surface — 2-complex being a continuous image $f(C)$ of a cylinder $C = S^1 \times [0,1]$ such that $\Gamma_1 = f(S^1 \times \{0\})$ and $\Gamma_2 = f(S^1 \times \{1\})$ and in the neighbourhood of every vertex of Γ_i the image of the corresponding boundary component lies in the union of the opposite half-edges.

Slice Knots and the Statement of the Problem

Definition

A free knot is called slice if it is cobordant to the trivial knot.

In other words, a slice knot can be spanned by a disc.

The **main question** is: how one can understand whether a knot is slice looking at its diagram?

Example

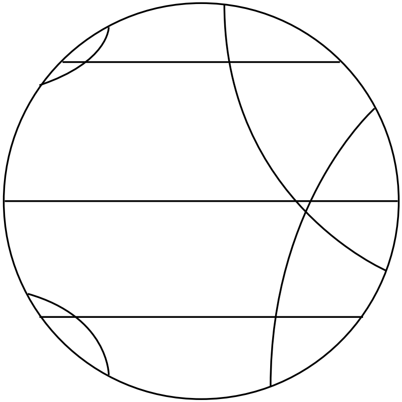

Figure: The first found example of a non-slice free knot

The Main Theorem on Cobordisms

The following theorem gives an answer to the sliceness problem for *odd* free knots.

Theorem (V.M.)

If a chord diagram of a free knot is odd, it is slice if and only there exists a pairing of its chords with no intersections.

This theorem translates a difficult topological question of sliceness to a purely combinatorial problem of chord pairing for a fairly large family of free knots.

Example 1

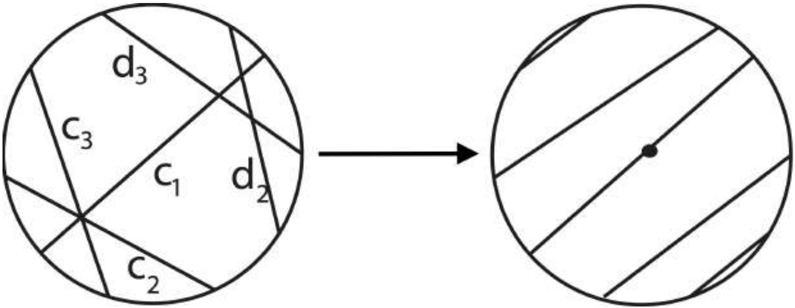

Figure: Correct pairing yields sliceness

Example 2

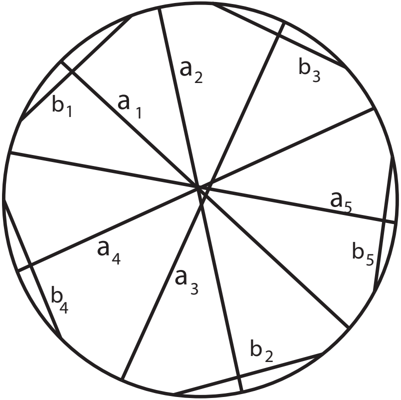

Figure: This free knot is not slice

The Notion of a 2-Knot

Definition

A 2-knot (resp. an n-component 2-link) is a smooth embedding in general position of a 2-sphere S^2 (resp. disjoint union of n spheres) into $\mathbb{R}^4$ or S^4 up to isotopy.

If one takes S_g instead of S^2, one obtains a *surface knot* (or *surface link* in case of many components).

Definition

A diagram of a 2-knot K in $\mathbb{R}^3$ is a projection in a general position of K in $\mathbb{R}^4$ to a subspace $\mathbb{R}^3$.

Two diagrams represent the same knot if and only if they can be related by a finite sequence of *Roseman moves*.

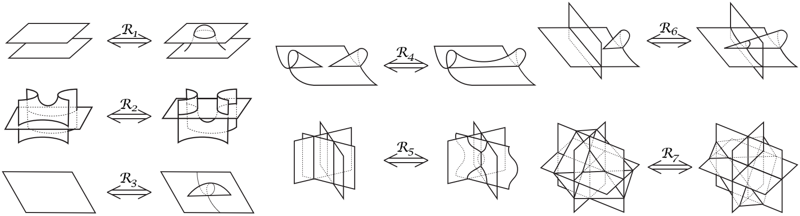

Figure: Roseman Moves

2-Knots Diagrams: Spherical

Definition

A *spherical diagram* (or Gauss diagram) is a 2-complex consisting of a sphere S and a set D of marked curves on it such that:

1. Every curve is either closed or ends with a cusp, the number of cusps is finite;
2. Every curve of the set D is paired with exactly one curve of that set, one of the paired curves is marked as *upper*, both curves are oriented (marked with arrows);
3. Two curves ending in the same cusp are paired and both arrows either look towards the cusp or away from it;
4. If two curves intersect, the curves paired with them intersect as well (thus a triple point appears on the sphere S three times).

Roseman Moves on Spherical Diagrams

Roseman moves have natural analogs for spherical diagrams.

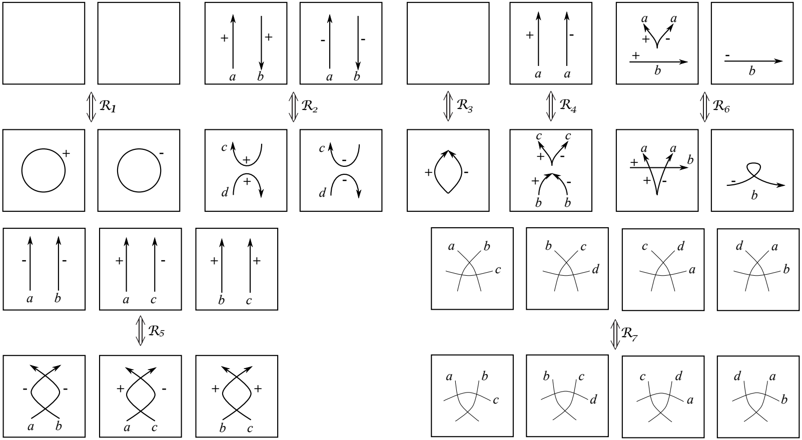

Figure: Roseman Moves for Spherical Diagrams

Parity Axioms

Consider the following system of axioms:

1. *Continuity axiom.* Parity is constant along double lines.
2. *Loop axiom.* A double line being an edge of a cylinder over a loop is even.
3. *Bigon axiom.* The sum of parities of two double lines being the edges of a cylinder over a bigon equals $0 \mod 2$.
4. *Triangle axiom.* The sum of parities of three double lines being the edges of a cylinder over a triangle equals $0 \mod 2$.
5. *Correspondence axiom.* The parities of the corresponding boundary double points are the same.

We can define *parity* in the following way:

Definition

Let $\mathcal{L}$ be a class of 2-links in $\mathbb{R}^3$, and let A be the set of double line of their diagrams. A mapping $P : A \to \mathbb{Z}_2$ is called parity if it satisfies the axioms 1–5.

Smoothings

In 1-dimensional knot theory *smoothing* of a crossing means cutting out a crossing with its small neighbourhood and reglueing two pairs of half-edges in one of the possible two ways:

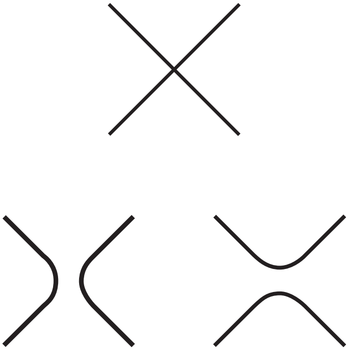

Figure: Smoothings of a crossing

Smoothings (cont.)

In 2-dimensional case not crossings, but double lines are smoothed. That is, locally one can think of a 2-dimensional smoothing as a 1-dimensional smoothing on a transversal section, multiplied by an interval. The trick is to define smoothing of a whole double line (or a family of those) in a compatible way. Here we present one approach, using spherical diagrams.

Consider a pair of paired curves γ, γ' on a spherical diagram of a knot. Cut the diagram along those lines. The resulting multi-component complex has four boundary components: γ_1, γ_1' on the sphere, and γ_2, γ_2' on the cut out pieces.

Now we glue those curves back together (respecting their orientation and intersecting curves) following the rule: non-prime curve must be glued to a prime one.

Smoothings and Triple Points

To complete our smoothing procedure we need to understand, what happens to the triple points the smoothed line goes through.
It is easy to see that the reglueing of γ's naturally induces a (1-dimensional) smoothing near the third preimage of a triple point involved. The type of this smoothing exactly corresponds to the choice of pairing between γ_i and γ'_j.

That procedure defines a *smoothing* of a double line. If we need to smoothen a set of double lines, we should do it one by one in some order. Note, that we don't claim that the result is independent of the order of smoothings. It may well be true but wasn't studied in detail yet.

Smoothing Lemma

There are two ways to smoothen a double line — just like in the 1-dimensional case. In case of a spherical diagram one of those to smoothings always produces an additional connected component, while the other one transforms the sphere into a sphere again (with a different set of curves).
Every smoothing reduces the number of curves on the spherical diagram. Thus, smoothing *all* double lines of a diagram (in any order) and every time choosing the smoothing that preserves the number of connected components we come to an empty spherical diagram. Thus we get the following lemma:

Lemma (D.F.)

For every free 2-knot diagram there exists a smoothing yielding a trivial knot.

Proof of the Theorem

Now we are ready to prove the main theorem.

One implication stated is fairly obvious: if a chord pairing exists, this pairing exactly describes the double lines of the spanning disc. Thus the knot is indeed slice.

Now we come to the tough part: if a knot with an odd chord diagram is slice, why there exists a correct chord pairing?
Consider an odd diagram K and spanning disc D. This complex may be regarded as a 2-knot with boundary.
This complex has double lines of two types: "inner" double lines and double lines ending at the boundary. Note that the smoothing process, described above, can be verbatim generalised to the case of a knot with boundary for inner double lines.
Let us smoothen all inner double lines.

Proof of the Theorem (cont.)

Due to the Smoothing Lemma the Gauss diagram of the resulting knot is a disc with some curves with their ends on the boundary. Since the original knot diagram K was odd in Gaussian sense, those double lines are odd as well (since the diagram didn't change in a small neighbourhood of its boundary).

Therefore, we have neither cusps nor triple points on the resulting diagram.

That means that every double line begins and ends on the knot boundary. So the crossings of the diagram K are paired — without intersections (due to the absence of triple points). Thus the claim is verified.

Some Open Problems and Work in Progress I

- How one can construct an invariant parity bracket for 2-knots, surface knots, links?
- There are different ways to define a smoothing of a double line. Which of them produce a reasonable 2-complex? How can they be used?
- Consider two knot diagrams. Is there a way to say if those free knots are cobordant looking just at their diagrams? For some particular classes of diagrams, maybe?
- How can one calculate a slice genus looking at a decorated 4-valent graph?
- Let us call a knot *elementary slice* if it can be spanned by a disc without cusps and triple points. In the same way one can define an elementary slice genus.
  Is it true that slice genus of a knot is bounded by g if and only if the elementary slice genus of this knot is bounded by g?

Some Open Problems and Work in Progress II

- Smoothing lemma generalisation: is it true that every knot of genus g can be smoothened to a trivial knot of genus at most g?
- Is it possible to get a nice criterion of sliceness for odd diagrams — say in the language of incidence matrices?
- Is it true (again, possibly for some family of diagrams) that if two free knots are cobordant, they are cobordant with no cusps and triple points? This question relates "dynamic" and "static" problems.

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