Three-points Kontsevich integral for braids

V. P. Leksin

State Social-Humanity University, Kolomna, Russia

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Let $\nabla = d - \Omega$ be a formal connection. Formal differential 1-form

$$\Omega = \sum_{\{i, j\} \subset \{1, 2, \ldots, n\}} B_{ij} \frac{d(z_i - z_j)}{z_i - z_j}$$

is the meromorphic 1-form on $\mathbb{C}^n$ with formal coefficients $B_{ij}$. Forms $\omega_{ij} = \frac{d(z_i - z_j)}{z_i - z_j}$ depend only unordered pair points $z_i$ and $z_j$. We suppose that $B_{ij} = B_{ji}$. As well-known constant formal coefficients $B_{ij}$ may be interpreted as chord diagrams.
Integrability of formal connections

Frobenius condition of the integrability

\[ d\Omega = \Omega \wedge \Omega \]

of formal connection \( \nabla \) is equivalent to two equalities \( d\Omega = 0 \) and \( \Omega \wedge \Omega = 0 \) or to the following relations

1. \[ [B_{ij}, B_{kl}] = 0, \text{ for } \{i, j\} \cap \{k, l\} = \emptyset; \quad (R1) \]

2. \[ [B_{ij}, B_{jk} + B_{ik}] = 0, \text{ for } i \neq j \neq k. \quad (R2) \]

Here \([A, B] = AB - BA\). Second series of relations (R2) coincide with 4-term relations for chord diagrams \( B_{ij} \).
Parallel transport of formal connections and Kontsevich integral

Basic ingredient of Kontsevich integral is the parallel transport

$$T_{\nabla}(\gamma) = 1 + \int_{\gamma} \Omega + \int_{\gamma} \Omega \Omega + \cdots$$

doing integrable formal connection $\nabla = d - \Omega$. Here $\gamma$ is a path in $\mathbb{C}^n \setminus \bigcup_{i \neq j} \{z_i \neq z_j\}$ presenting a braid. Summands $\int_{\gamma} (\Omega)^r, r = 1, 2, \ldots$ are the $r$-iterated Chen integrals

$$\int_{\gamma} (\Omega)^r = \sum_{p_1 < q_1, p_2 < q_2 \ldots p_r < q_r} B_{p_1, q_1} B_{p_2, q_2} \cdots B_{p_r, q_r} \int_{\gamma} \omega_{p_1, q_1} \omega_{p_2, q_2} \cdots \omega_{p_r, q_r},$$
where

\[ \int_{\gamma} \omega_{p_1,q_1} \omega_{p_2,q_2} \cdots \omega_{p_r,q_r} = \]

\[ = \int_{\Delta^r} f_1(t_1)f_2(t_2) \cdots f_r(t_r) dt_1 dt_2 \cdots dt_r = \]

\[ = \int_{0}^{1} (\ldots (\int_{0}^{t_4} (\int_{0}^{t_2} f_1(t_1) dt_1) f_2(t_2) dt_2) dt_3 \ldots) f_r(t_r) dt_r \]

Here

\[ \Delta^r = \{(t_1, t_2, \ldots, t_r) | 0 \leq t_1 \ll t_2 \leq \ldots \leq t_r \leq 1 \subset \mathbb{R}^r\} = \]

is r-simplex in affine space \( \mathbb{R}^r \) and

\[ \gamma^* \omega_{p_s,q_s} = f_s(t) dt \]

is differential 1-form on segment \( I = [0, 1] \).
The value of the parallel transport $T_\nabla(\gamma)$ of integrable connection $\nabla$ depends only from the homotopic class of the path $\gamma$ with fixed ends.

Parallel transport $T_\nabla$ possesses the multiplicative property

$$T_\nabla(\gamma_1\gamma_2) = T_\nabla(\gamma_1)T_\nabla(\gamma_2),$$

when the product $\gamma_1\gamma_2$ is defined.
Representations of pure braid groups in chord diagram algebras

Let $Ch_n = \mathbb{C}[B_{ij}, 1 \leq i \neq j \leq n]/J$ be chord diagram algebra. Ideal $J$ in the polynomial algebra of non-commutative variables $\mathbb{C}[B_{ij}, 1 \leq i \neq j \leq n]$ is generated by terms $[B_{ij}, B_{kl}], \{i, j\} \cap \{k, l\} = \emptyset; [B_{ij}, B_{jk} + B_{ik}] i \neq j \neq k$. Denote $\mathbb{C}_*^n$ the complement $\mathbb{C}_*^n = \mathbb{C}^n \setminus \bigcup_{1 \leq i < j \leq n} \{z_i - z_j = 0\}$ of the $\mathbb{C}^n$ to the union of diagonal hyperplanes $z_i - z_j = 0$. Take $z_0 = (1, 2, \ldots, n) \in \mathbb{C}_*^n$. The restriction of the parallel transport $T_\nabla$ of integrable connection $\nabla$ on the loop space $\Omega_{z_0} \mathbb{C}_*^n$ defines a representation $\rho_\nabla$ of the fundamental group $\pi_1(\mathbb{C}_*^n, z_0)$. 
This fundamental group is isomorphic to the pure braid group $P_n$.

We obtain the representation

$$\rho_\triangledown : P_n \rightarrow \widehat{Ch}_n$$

in completed chord diagram algebra $\widehat{Ch}_n = C << B_{ij}, 1 \leq i \neq j \leq n >> /J$.

Since $Ker \rho_\triangledown = 1$ (T.Kohno) then $\rho_\triangledown$ is full invariant of pure braids with values in the algebra $\widehat{Ch}_n$. 
Three-point analog of the Kontsevich integral

V.O. Manturov proposed to consider the formal connections with forms

$$\Omega_1 = \sum_{\{i,j,k\}\subset\{1,2,\ldots,n\}} A_{ijk} \omega_{ijk},$$

where meromorphic 1-forms

$$\omega_{ijk} = \frac{d(z_i - z_j)}{z_i - z_j} + \frac{d(z_j - z_k)}{z_j - z_k} + \frac{d(z_i - z_k)}{z_i - z_k}$$

depend only from unordered collection of three points $z_i, z_j, z_k$ and $A_{ijk}$ are formal coefficients with unordered collections of subscripts.
The integrability condition of this connection may be obtained by rewriting of the form \( \Omega_1 \) as \( \Omega_1 = \sum_{\{i, j\} \subset \{1, 2, ..., n\}} B_{ij} \omega_{ij} \). Here

\[
B_{ij} = \sum_{r=1, r \neq i, j}^{n} A_{ijr}.
\]

Then relations on \( A_{ijk} \) (threesome - in Manturov terminology) are equalities

\[
\left[ \sum_{r=1, r \neq i, j}^{n} A_{ijr}, \sum_{m=1, m \neq k, l}^{n} A_{klm} \right] = 0, \{i, j\} \cap \{k, l\} = \emptyset. \quad (R3)
\]

\[
\left[ \sum_{p=1, p \neq i, j}^{n} A_{ijp}, \sum_{q=1, q \neq j, k}^{n} A_{klq} + \sum_{r=1, r \neq i, k}^{n} A_{klr} \right] = 0, i \neq j \neq k. \quad (R4)
\]
In particular case $n = 4$ we have relations:

\[
\begin{align*}
[A_{123} + A_{124}, A_{134} + A_{234}] &= 0; \\
[A_{123} + A_{134}, A_{124} + A_{234}] &= 0; \\
[A_{124} + A_{134}, A_{123} + A_{234}] &= 0; \\
[A_{123} + A_{124}, 2A_{123} + A_{134} + A_{234}] &= 0; \\
[A_{123} + A_{124}, 2A_{124} + A_{134} + A_{234}] &= 0, \\
[A_{123} + A_{234}, 2A_{234} + A_{124} + A_{134}] &= 0; \\
[A_{123} + A_{234}, 2A_{123} + A_{134} + A_{234}] &= 0.
\end{align*}
\]
Representation of the $P_n$ in threesecant algebra

Define the threesecatnt algebra as quotient of algebra non-commutative polynomials

$$TS_n = \mathbb{C}[A_{ijk}, \{i, j k\} \subset \{1, 2, \ldots, n\}]/J_1,$$

where $J_1$ is ideal generated right sides of relations $(R_1)$ and $(R_2)$. The completed the threesecatnt algebra is following quotient-algebra

$$\widehat{TS}_n = \mathbb{C} \ll A_{ijk}, \{i, j k\} \subset \{1, 2, \ldots, n\} \gg /J_1,$$

Parallel transport $T_{\nabla_1}$ of the integrable connection $\nabla_1 = d - \Omega_1$ with form $\Omega_1$ under restriction on the loop space $\Omega_{z_0} \mathbb{C}^n_*$ defines
a representation $\rho_{\nabla_1}$ of pure braid group $P_n$ in the completed three-secant algebra, i.e the associative algebra of series from non-commutative variables $A_{ijk}$ with relations $(R_1)$ and $(R_2)$

$$\rho_{\nabla_1} : P_n \rightarrow \widehat{TS}_n.$$ 

Let a path $\gamma \in \Omega_{z_0} \mathbb{C}^n_*$ defines a braid $b \in P_n$. We call the value $T_{\nabla_1}(\gamma) = \rho_{\nabla_1}(b)$ the three-point Kontsevich integral for braid $b$. 
Generators and relations of pure braid group $P_n$

Let $L$ be some line in affine space $\mathbb{C}^n$, intersecting all hyperplanes $H_{ij} = \{z \in \mathbb{C}^n | z_i = z_j\}$ in general position. Denote $p_{ij} = l \cap H_{ij} \in L$ points of intersection $L$ and $H_{ij}$, $1 \leq i < j \leq n$. Take loops $\gamma_{ij}$ on $L$ with initial point $p \neq p_{ij}, 1 \leq i < j \leq n$, $l$ single around about $p_{ij}$. We suppose that loops $\gamma_{ij} \cap \gamma_{rs} = p$, $\{i, j\} \neq \{r, s\}$. Loops $\gamma_{ij}, 1 \leq i < j \leq n$ present a system of free generators $b_{ij}$ in free group $\pi_1(L \setminus \{p_{ij}, 1 \leq i < j \leq n\}, z^0)$ and a system of generators $b_{ij}$ in pure braid groups $PB_n = \pi_1(\mathbb{C}^n_*, z_0 = (1, 2, \ldots, n))$. Relations in $PB_n$ correspond loops in a transversal plane to the line $L$. These loops go around about points of intersection of hyperplanes $H_{ijk}$ and $H_{ijkl}$. 
Generators and relations of Birman-Ko-Lee:

\[ b_{ij}b_{kl} = b_{kl}b_{ij}, \quad i < j < k < l \text{ или } i < k < l < j, \]

\[ b_{ij}b_{ik}b_{jk} = b_{ik}b_{jk}b_{ij}, \quad i < j < k \]

\[ b_{jl}b_{kl}b_{ik}b_{jk} = b_{jk}b_{ik}b_{kl}b_{jl}, \quad i < j < k < l. \]

Artin generators and Birman-Ko-Lee generators

\[ b_{ij} = A_{i,i+1}A_{i+1,i+2} \cdots A_{j-1,j}. \]
Manturov group $G_n^3$

The Manturov group $G_n^3$ is defined as a group with generators $a_{ijk}, \{i, j, k\} \subset \{1, 2, \ldots, n\}$ and relations

$$a_{ijk}^2 = 1,$$

$$a_{ijk}a_{pqr} = a_{pqr}a_{ijk}, \text{ if } |\{i, j, k\} \cap \{p, q, r\}| \leq 1,$$

$$(a_{jkl}a_{ikl}a_{ijl}a_{ijk})^2 = 1, \text{ for } \forall\{i, j, k, l\} \subset \{1, 2, \ldots, n\}.$$
Manturov-Nikonov representation of pure braid groups

Manturov-Nikonov defined the homomorphism $\varphi_n$ of the pure braid $P_n$ in the group $G^3_n$ by values on generators $b_{ij}$

$$\varphi_n(b_{ij}) = c_{ii+1}^{-1} \cdots c_{ij-1}^{-1} c_{ij} c_{ij-1} \cdots c_{ii+1},$$  \hspace{1cm} (1)

где

$$c_{ij} = \prod_{k=j+1}^{n} a_{ijk} \prod_{k=1}^{j-1} a_{ijk}.$$  \hspace{1cm} (2)
Realization of $\varphi_n(b_{ij})$ as values parallel transport

Let $\gamma_{ij} \in \Omega_{z_0} \mathbb{C}_*^n$ be loops presenting generators $b_{ij}$ of the $P_n$. There are exist a thresedants $A_{ijk}$ in the group algebra $\mathbb{C}[G_n^3]$ is the group algebra of the Manturov group $G_n^3$, such that we have equalities $T_{\nabla_1}(\gamma_{ij}) = \varphi_n(b_{ij})$. 
Lappo-Danilevskii inversion method

Let us consider the group algebra $g_n^3 = \mathbb{C}[G_n^3]$ as Lie algebra with respect ordinary Lie bracket $[a, b] = ab - ba$. Universal enveloping algebra $U(g_n^3)$ is isomorphic to $\mathbb{C}[G_n^3]$. Let $\mathbb{C}[\hat{G}_n^3]$ be the completion of the $\mathbb{C}[G_n^3]$ by the augmentation ideal $\mathfrak{n} J \subset \mathbb{C}[G_n^3]$, generated elements $a g - 1, g \in G_n^3$. We will look for $B_{ij}$ in the ideal $\hat{J} \subset \mathbb{C}[\hat{G}_n^3]$.

Consider the system of equations

$$\varphi_n(b_{ij}) = T_\nabla(\gamma_{ij})$$
or

\[ c_{i,i+1}^{-1} \cdots c_{i,j-1}^{-1} c_{ij}^2 c_{i,j-1} \cdots c_{i,i+1} = 1 + \int_{\gamma_{ij}} \Omega + \int_{\gamma_{ij}} \Omega \Omega + \cdots + \int_{\gamma_{\gamma_{ij}}} \Omega^m + \cdots. \]

Rewritten last system

\[ c_{i,i+1}^{-1} \cdots c_{i,j-1}^{-1} (c_{ij}^2 - 1) c_{i,j-1} \cdots c_{i,i+1} = \sum_{p<q} B_{pq} \int_{\gamma_{ij}} \omega_{pq} + \sum_{p<q, r<s} B_{pq} B_{rs} \int_{\gamma_{ij}} \omega_{pq} \omega_{rs} + \cdots + \sum_{p_1<q_1, p_2<q_2 \ldots p_m<q_m} B_{p_1,q_1} B_{p_2,q_2} \cdots B_{p_m,q_m} \int_{\gamma_{ij}} \omega_{p_1,q_1} \omega_{p_2,q_2} \cdots \omega_{p_m,q_m} + \cdots. \]

Here \( \omega_{pq} = \frac{d(z_p-z_q)}{z_p-z_q} \).

By theorem about inverting formal series (then the matrix of the linear part is invertible) we obtain series for \( B_{pq}, p < q \) as series
from left parties $M_{ij} = c_{i,i+1}^{-1} \ldots c_{i,j-1}^{-1}(c_{ij}^2 - 1)c_{i,j-1} \ldots c_{i,i+1}$ of equations above.
Integrability of the formal connections

Integrability of the connection $\nabla = d - \Omega$ may extracted from generating commutator relations of the $P_n$.

We obtain the formal connection $\nabla_1 = d - \Omega_1$ solving the system of the linear equations

$$B_{ij} = \sum_{r=1, r \neq i, j}^{n} A_{ijr}.$$ 

The integrability of the $\nabla$ implies the integrability of the $\nabla_1$.

Thus we obtain realization

$$\varphi(b_{ij}) = \rho_{\nabla_1}(b_{ij})$$
of the representation $\varphi_n$ as the monodromy representation of a integrable connection.
References


1501.05208 [math.GT].
