Equivalence of nice Heegaard diagrams and combinatorial Floer homology

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Outline of talk

1. From Morse homology to Heegaard Floer homology
2. Combinatorial descriptions
3. Chain complex
4. Equivalence and invariance
A Morse function on a manifold is a real-valued function that looks like

\[ f(x_1, \cdots, x_n) = -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2 \]

near a critical point under some coordinate system.
A Morse function on the torus
Morse homology of the torus

So we have
\[ \partial a = \partial b = \partial c = \partial d = 0 \]

And the homology is
\[ H_n^{\text{Morse}}(T^2) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \end{cases} \]
Lagrangian intersection Floer homology

Given a symplectic manifold \((M, \omega)\) and two compact Lagrangian submanifolds \(L_0\) and \(L_1\), the Floer homology is roughly the Morse theory for the following action functional on the path space

\[
A : \widetilde{P}(L_0, L_1) \to \mathbb{R}, \quad A(\gamma, [u]) = \int u^* \omega
\]

The critical points of \(A\) are constant paths, or \(L_0 \cap L_1\). The Euler-Lagrange equation is the Cauchy-Riemann equation and the gradient flow lines are pseudo-holomorphic curves. The Floer homology \(HF(L_0, L_1)\) is generated by \(L_0 \cap L_1\) with the differential counting dimension one flow lines.

Give 3-manifold \(Y\) with Heegaard splitting along a surface \(\Sigma_g\), we can define a Lagrangian intersection Floer homology as follows: the space of flat connections on \(\Sigma_g\) is a symplectic manifold of dimension \(6g - 6\), and the flat connections on \(\Sigma_g\) that extends over each of the two handlebodies form a Lagrangian submanifold.
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Example of Lagrangian Floer homology

So we have

$$\partial a = \partial c = b$$

and its homology is $\textit{HF}(L_0, L_1) \cong \mathbb{Z}$, generated by $a + c$ or $a - c$. 
Heegaard splittings

Every closed orientable three-manifold $Y$ has an embedded surface which splits $Y$ into two handlebodies. Such a decomposition is called a Heegaard splitting.

The following is the standard genus one Heegaard splitting for $S^3$:

A Heegaard splitting is characterized by the Heegaard surface, together two sets of curves bounding disks in the two handlebodies. The above one is denoted by $(T_1, A, B)$. 
A genus two Heegaard splitting of the three-sphere
Heegaard Floer homology

Given a three-manifold $Y$ with a Heegaard splitting $\mathcal{H} = (\Sigma, \alpha, \beta, w)$, where $w \in \Sigma \setminus (\alpha \cup \beta)$ is a reference point. The $\text{Sym}^g(\Sigma - w)$ is a symplectic manifold, together with two Lagrangian tori

$$T_\alpha = \alpha_1 \times \cdots \times \alpha_g, \quad T_\beta = \beta_1 \times \cdots \times \beta_g$$

The Heegaard Floer homology $\widehat{HF}(Y)$ is defined as follows: the generators consist of $T_\alpha \cap T_\beta$, or equivalently,

$$x = (x_1, \cdots, x_g) \in (\alpha_1 \cap \beta_{\sigma(1)}) \times \cdots \times (\alpha_g \cap \beta_{\sigma(g)})$$

and the differential counts for index one holo disk from $x$ to $y$.

For a null homologous knot $K \subset Y$, $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$. $\widehat{HFK}(Y, K)$ is defined similarly, but in $\text{Sym}^g(\Sigma - w - z)$. 
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A genus two Heegaard splitting of the three-sphere
The Floer chain complex depends on the complex structure

To define holomorphic disks, we choose a complex structure on $\Sigma$. The chain complex may be different when the complex structure varies (though the quasi-isomorphism type does not change).

When $a < b$, we have

Note that the case $a = b$ is NOT generic.

Thus we see that the chain complex depends on the complex structure.
The Floer chain complex depends on the complex structure.

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When $a > b$, we have

\[ x_1y_1 \rightarrow x_3y_1 \rightarrow x_1y_3 \rightarrow x_3y_3 \]
\[ x_2y_1 \rightarrow x_1y_2 \rightarrow x_3y_2 \rightarrow x_2y_3 \]
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The Poincaré homology sphere $\Sigma(2, 3, 5)$

#Generators: 21, # Differentials: ???
A Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, w)$ is **nice** if every region is a bigon or square, except the (preferred) region containing $w$, which is a polygon. Regions are connected components of $\Sigma \setminus (\alpha \cup \beta)$.

Generators: $x = (x_1, \cdots, x_g) \in (\alpha_1 \cap \beta_{\sigma(1)}) \times \cdots \times (\alpha_g \cap \beta_{\sigma(g)})$

($\sigma$'s are elements in $S_g$.)

There are two types of differentials:
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\[
\begin{align*}
x & = (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_g) \\
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$y = (x_1, \cdots, y_i, \cdots, y_j, \cdots, x_g)$
Existence

Theorem (Sarkar-W)

Every closed orientable three-manifold admits a nice Heegaard diagram. Every null-homologous knot in a closed orientable three-manifold admits a nice Heegaard diagram.

Theorem (W)

Every pointed Heegaard diagram is isotopic to a nice Heegaard diagram.
**Theorem (Sarkar-W)**

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Example: the trefoil knot
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Equivalence and Invariance
Example: the trefoil knot, computation

\[ s(x) - s(y) = n_z(\phi) - n_w(\phi) \]

\[ gr(x) - gr(y) = 1 - 2n_w(\phi) \]

The chain complex is

\[ \hat{HC}(S^3, T) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad \hat{HC}(S^3) = \mathbb{Z} \]
Example: the Poincaré homology sphere $\Sigma(2, 3, 5)$

#Generators: 21, # Differentials: ???
Example: the Poincaré homology sphere - a nice diagram

#Generators: 335,  #Differentials: 505,  $\widehat{HC}(\Sigma(2,3,5)) = \mathbb{Z}$
Every knot in $S^3$ has a grid diagram, which is a multiply-pointed genus one Heegaard diagram of $S^3$. 

![Grid Diagram](image)
Ozsváth and Szabó used convenient diagrams, which is a special kind of nice diagrams to define the hat Heegaard Floer homology, and showed the invariance.
Theorem (W)

Any two nice Heegaard diagrams for a closed oriented three-manifold can be transformed to one another via admissible moves.

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Equivalence and Invariance

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Admissible move, isotopy

Here $D'_1$ and $D'_2$ either a bigon or the preferred region $D_w$. 

(a) \hspace{5cm} (b)
Admissible move, isotopy

Here $D'_2$ and $D'_3$ are either a bigon or the preferred region $D_w$. 
An *admissible stabilization* is a stabilization in a small neighborhood of the marked point \( w \), followed by a finger move of the new beta curve to a bigon or \( D_w \).
Admissible move, handleslide and stablization

An *admissible stabilization* is a stabilization in a small neighborhood of the marked point $w$, followed by a finger move of the new beta curve to a bigon or $D_w$. 
Why a chain complex? index two disks.

Let $y$ be a generator appearing in $\partial^2 x$, i.e., there is a index two disk connecting $x$ to $y$. It will looks like (let us just consider squares, for simplicity)
Why a chain complex? “Gromov compactness”

We see that the generator $y$ (white dot) appears in $\partial^2 x$ in pairs. So $\partial^2 x = 0$ with $\mathbb{Z}_2$ coefficients.
Again the generator $y$ (white dot) appears in $\partial^2 x$ in pairs.
Why a chain complex? “Gromov compactness”, continued
Proof of Equivalence, admissible handleslides

**Proposition**

A handleslide on a Heegaard diagram can be made admissible modulo admissible isotopies.
Proposition

Let $\mathcal{H} = (\Sigma, \alpha, \beta, \gamma w)$ be a pointed triple diagram. Suppose both $\mathcal{H}^1 = (\Sigma, \alpha, \beta, w)$ and $\mathcal{H}^2 = (\Sigma, \alpha, \gamma, w)$ are nice diagrams and the beta and gamma curves are isotopic in the complement of $w$. Then $\mathcal{H}^1$ and $\mathcal{H}^2$ can be made identical after admissible moves and ambient isotopy of $\Sigma$. 
Proof of equivalence

Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are two nice diagrams for $Y$.

- They become equivalent after some admissible moves.
- Make the alpha curves isotopic in $\Sigma \setminus w$ after admissible handleslides.
- Make the two set of alpha curves identical.
- By admissible handleslides of beta curves, make beta and gamma curves isotopic in $\Sigma \setminus w$.
- Make beta and gamma curves identical after admissible isotopies.
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Algebraic lemma

Let \((C, \partial)\) be a graded chain complex generated by \(G = \{g_1, \cdots, g_m\}\) and the differential \(\partial\) is of degree \(-1.\) We write

\[
\partial g_i = \sum_{j=1}^{m} a^j_i g_j.
\]

Suppose \(a^l_k = 1.\) Let \(C'\) be the vector space generated by \(G \setminus \{g_k, g_l\}\) with the same degree as in \(C.\) Define

\[
\partial'(g_i) := \sum_{j \neq k, l} (a^j_i + a^l_i a^j_k) g_j.
\]

Let \(\Phi : C \to C', \quad \Phi(g_k) := 0, \quad \Phi(g_l) := \sum_{j \neq k, l} a^j_k g_j, \quad \Phi(g_j) := g_j \text{ (for } j \neq k, l)\)

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\Psi : C' \to C, \quad \Psi(g_j) = g_j + a^j_l g_k \quad (j \neq k, l).
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Lemma

\((C', \partial')\) is a chain complex (that is, \(\partial' \circ \partial' = 0).\) Moreover, \((C, \partial)\) and \((C', \partial')\) are chain equivalent under the pair \((\Phi, \Psi).\)
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**Lemma**

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Proof of Invariance

Proposition (Handleslide invariance)

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ are two nice diagrams which differ by an admissible handleslide. Then $\widetilde{CC}(\mathcal{H}_1)$ and $\widetilde{CC}(\mathcal{H}_2)$ are chain equivalent.
Equivalence and Invariance
Proposition (isotopy invariance)

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ are two nice diagrams which differ by an admissible isotopy. Then $\widehat{CC}(\mathcal{H}_1)$ and $\widehat{CC}(\mathcal{H}_2)$ are chain equivalent.
Thank you!

спасибо!
Я помню чудное мгновенье:
Передо мной явилась ты,
Как мимолетное виденье,
Как гений чистой красоты.