Equivalence of nice Heegaard diagrams and combinatorial Floer homology

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Outline of talk

1. From Morse homology to Heegaard FLoer homology
2. Combinatorial descriptions
3. Chain complex
4. Equivalence and invariance

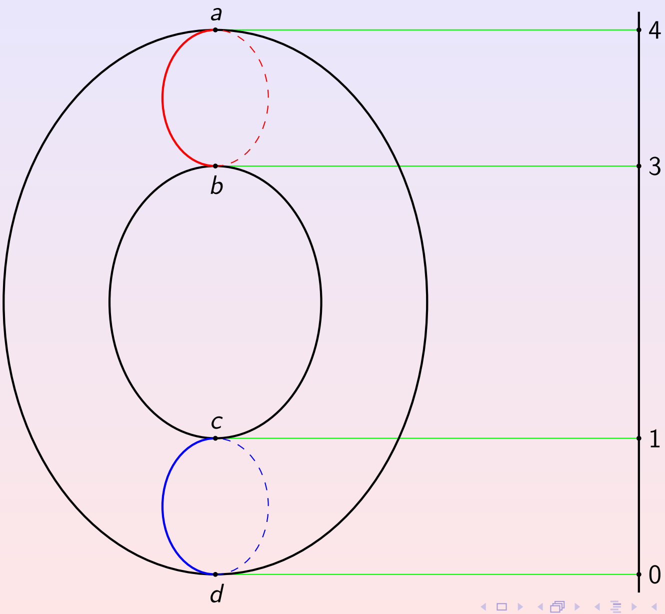
Morse function

A Morse function on a manifold is a real-valued function that looks like

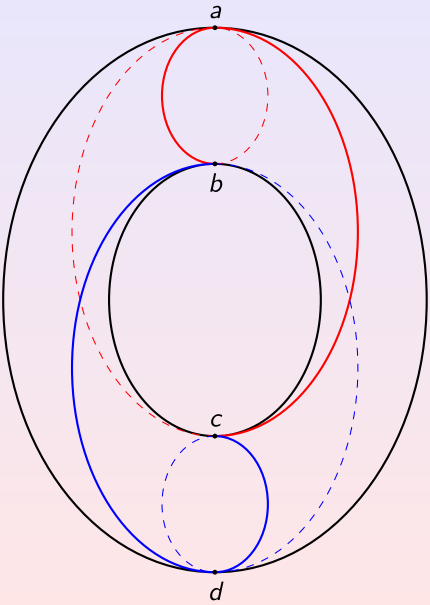
$$f(x_1, \cdots, x_n) = -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$$

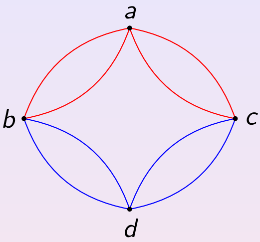
near a critical point under some coordinate system.

A Morse function on the torus



Morse homology of the torus





So we have

$$\partial a = \partial b = \partial c = \partial d = 0$$

And the homology is

$$H_n^{Morse}(T^2) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \end{cases}$$

Lagrangian intersection Floer homology

Given a symplectic manifold (M, ω) and two compact Lagrangian submanifolds L_0 and L_1, the Floer homology is roughly the Morse theory for the following action functional on the path space

$$A : \widetilde{\mathcal{P}}(L_0, L_1) \to \mathbb{R}, \quad A(\gamma, [u]) = \int u^*\omega$$

The critical points of A are constant paths, or $L_0 \cap L_1$. The Euler-Lagrange equation is the Cauchy-Riemann equation and the gradient flow lines are pseudo-holomorphic curves. The Floer homology $HF(L_0, L_1)$ is generated by $L_0 \cap L_1$ with the differential counting dimension one flow lines.

Give 3-manifold Y with Heegaard splitting along a surface Σ_g, we can define a Lagrangian intersection Floer homology as follows: the space of flat connections on Σ_g is a symplectic manifold of dimension $6g - 6$, and the flat connections on Σ_g that extends over each of the two handlebodies form a Lagrangian submanifold.

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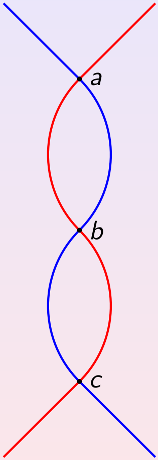
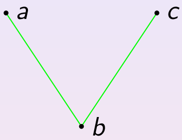
Example of Lagrangian Floer homology





So we have

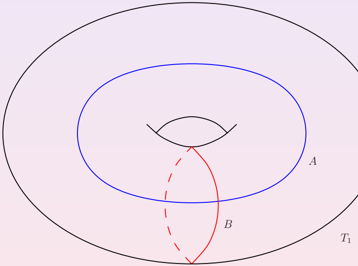
$$\partial a = \partial c = b$$

and its homology is $HF(L_0, L_1) \cong \mathbb{Z}$, generated by $a + c$ or $a - c$.

Heegaard splittings

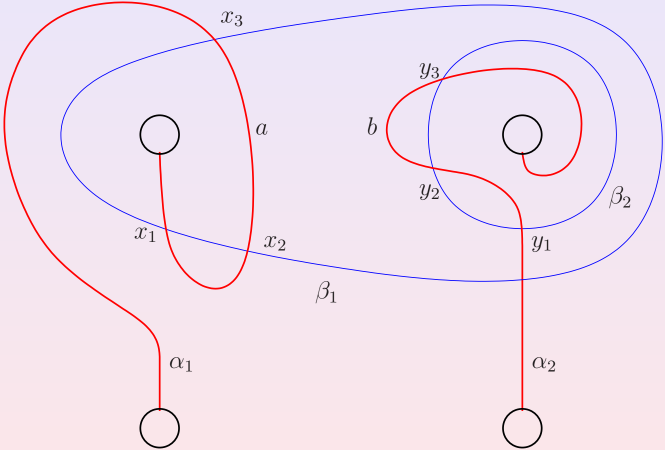
Every closed orientable three-manifold Y has an embedded surface which splits Y into two handlebodies. Such a decomposition is called a Heegaard splitting.

The following is the standard genus one Heegaard splitting for S^3



A Heegaard splitting is characterized by the Heegaard surface, together two sets of curves bounding disks in the two handlebodies. The above one is denoted by (T_1, A, B).

A genus two Heegaard splitting of the three-sphere



Heegaard Floer homology

Given a three-manifold Y with a Heegaard splitting $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$, where $w \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ is a reference point. The $\mathrm{Sym}^g(\Sigma - w)$ is a symplectic manifold, together with two Lagrangian tori

$$T_\alpha = \alpha_1 \times \cdots \times \alpha_g, \quad T_\beta = \beta_1 \times \cdots \times \beta_g$$

The Heegaard Floer homology $\widehat{HF}(Y)$ is defined as follows: the generators consist of $T_\alpha \cap T_\beta$, or equivalently,

$$\boldsymbol{x} = (x_1, \cdots, x_g) \in (\alpha_1 \cap \beta_{\sigma(1)}) \times \cdots \times (\alpha_g \cap \beta_{\sigma(g)})$$

and the differential counts for index one holo disk from $\boldsymbol{x}$ to $\boldsymbol{y}$.

For a null homologous knot $K \subset Y$, $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$. $\widehat{HFK}(Y, K)$ is defined similarly, but in $\mathrm{Sym}^g(\Sigma - w - z)$.

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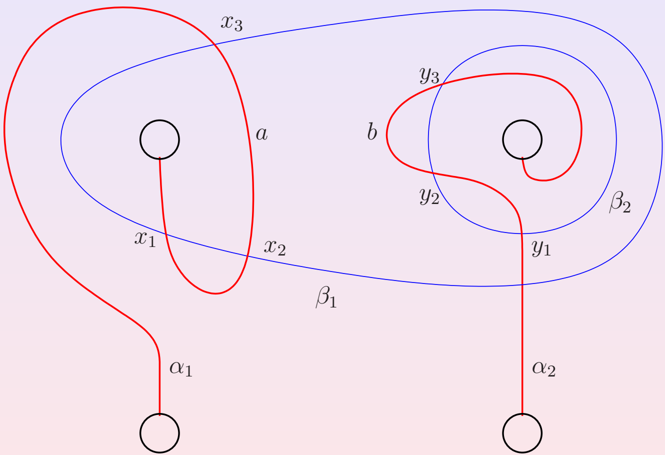
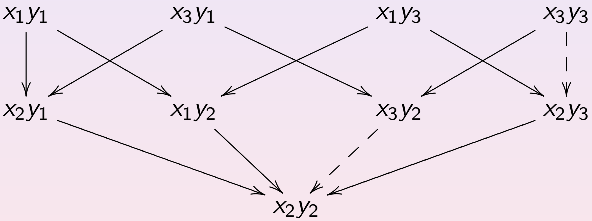


The Floer chain complex depends on the complex structure

To define holomorphic disks, we choose a complex structure on Σ. The chain complex may be different when the complex structure varies (though the quasi-isomorphism type does not change).

When $a < b$, we have



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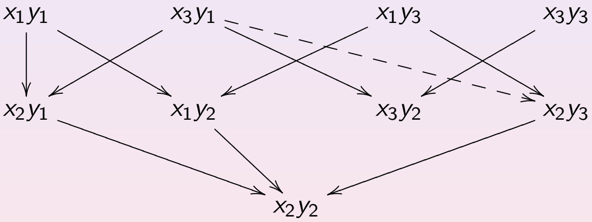

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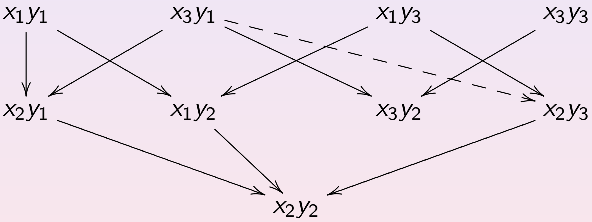

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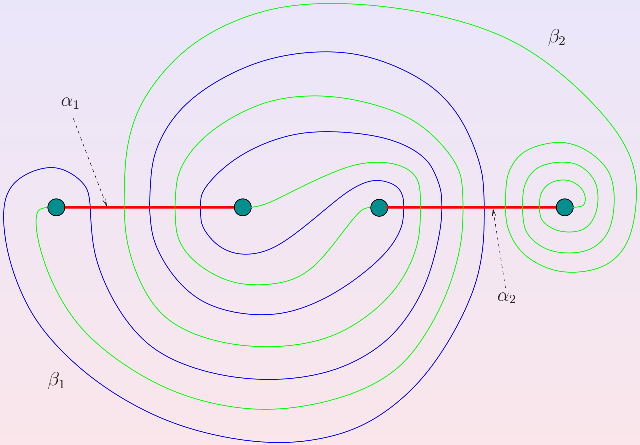
The Poincaré homology sphere $\Sigma(2,3,5)$



#Generators: 21, # Differentials: ???

Combinatorial Floer homology

Definition

A Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ is **nice** *if every region is a bigon or square, except the (preferred) region containing w, which is a polygon. Regions are connected components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$.*

Generators: $\boldsymbol{x} = (x_1, \cdots, x_g) \in (\alpha_1 \cap \beta_{\sigma(1)}) \times \cdots \times (\alpha_g \cap \beta_{\sigma(g)})$
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There are two types of differentials:

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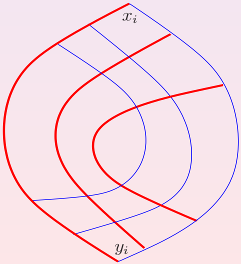

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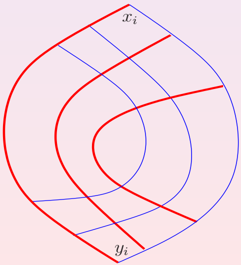

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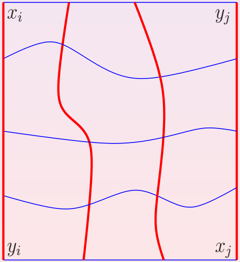

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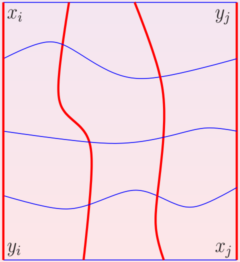

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Existence

Theorem (Sarkar-W)

Every closed orientable three-manifold admits a nice Heegaard diagram. Every null-homologous knot in a closed orientable three-manifold admits a nice Heegaard diagram.

Theorem (W)

Every pointed Heegaard diagram is isotopic to a nice Heegaard diagram.

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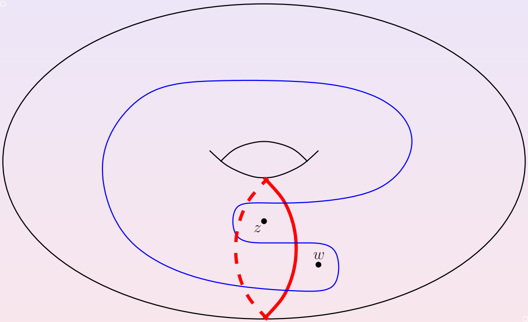

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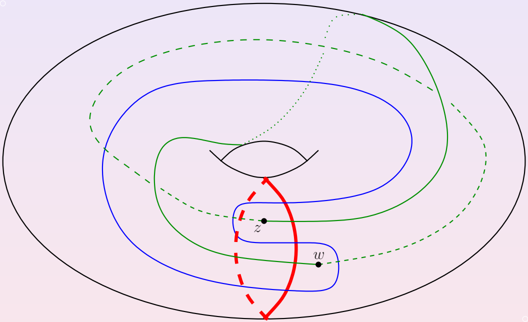

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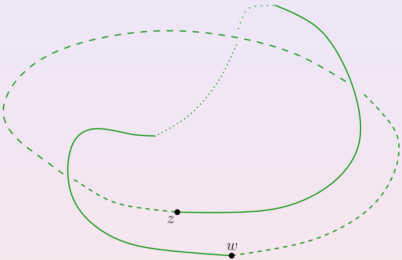

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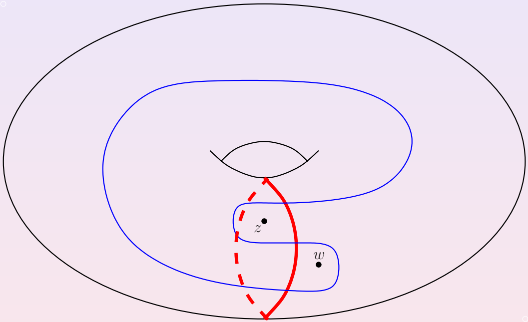
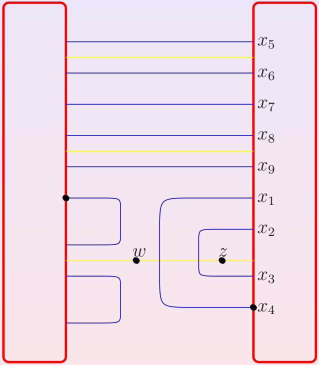


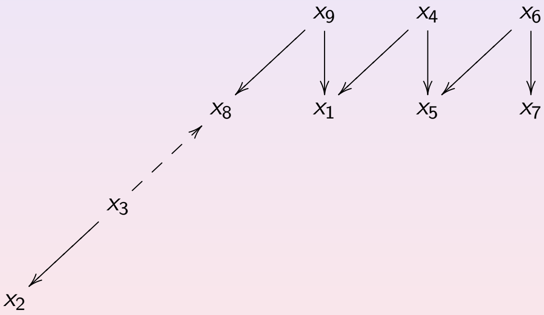
Example: the trefoil knot, computation



$$s(x) - s(y) = n_z(\phi) - n_w(\phi)$$

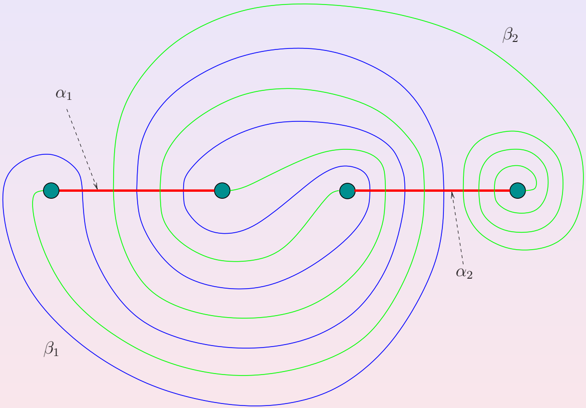
$$gr(x) - gr(y) = 1 - 2n_w(\phi)$$

The chain complex is



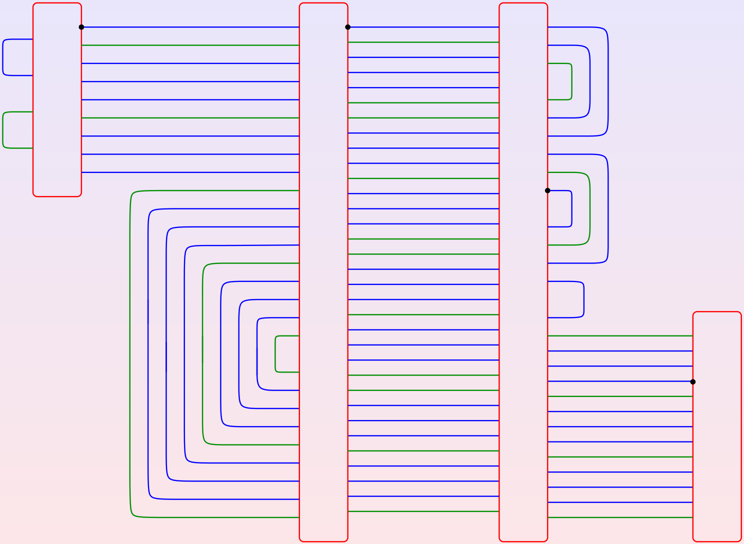
$$\widehat{HC}(S^3, T) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad \widehat{HC}(S^3) = \mathbb{Z}$$

Example: the Poincaré homology sphere $\Sigma(2, 3, 5)$



#Generators: 21, # Differentials: ???

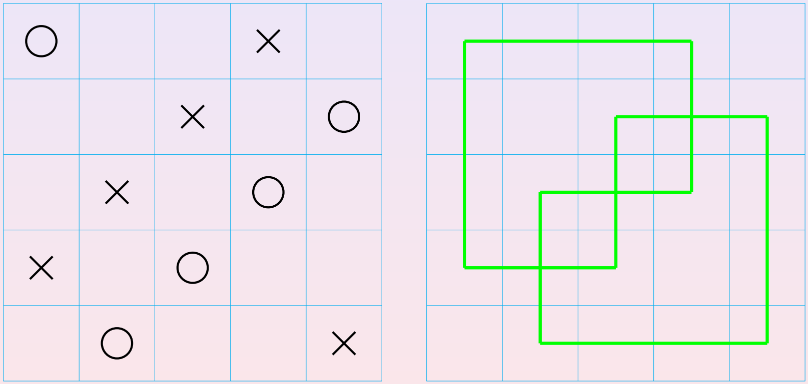
Example: the Poincaré homology sphere - a nice diagram



#Generators: 335,   #Differentials: 505, $\widehat{HC}(\Sigma(2,3,5)) = \mathbb{Z}$

Grid diagram and knot Floer homology

Every knot in S^3 has a grid diagram, which is a multiply-pointed genus one Heegaard diagram of S^3.



Convenient diagrams and invariance

Ozsváth and Szabó used convenient diagrams, which is a special kind of nice diagrams to define the hat Heegaard Floer homology, and showed the invariance.

Theorem (W)

Any two nice Heegaard diagrams for a closed oriented three-manifold can be transformed to one another via admissble moves.

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For a given closed oriented three-manifold, the Floer homology does not depend on the choice of the nice Heegaard diagram.

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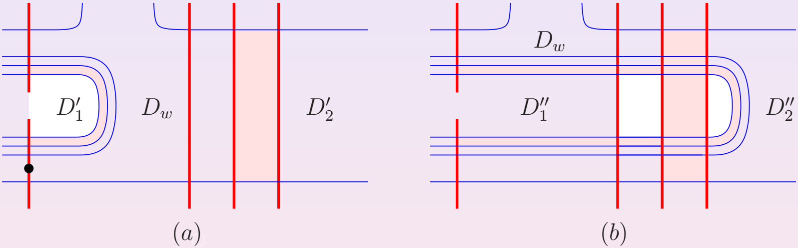
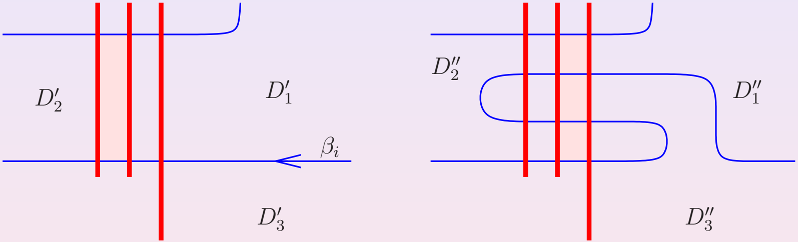
Admissible move, isotopy



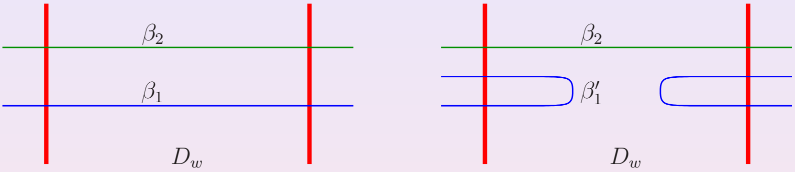
Here D_1' and D_2' either a bigon or the preferred region D_w.

Admissible move, isotopy



Here D_2' and D_3' are either a bigon or the preferred region D_w.

Admissible move, handleslide and stablization



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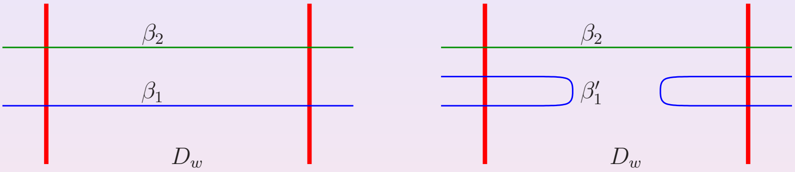

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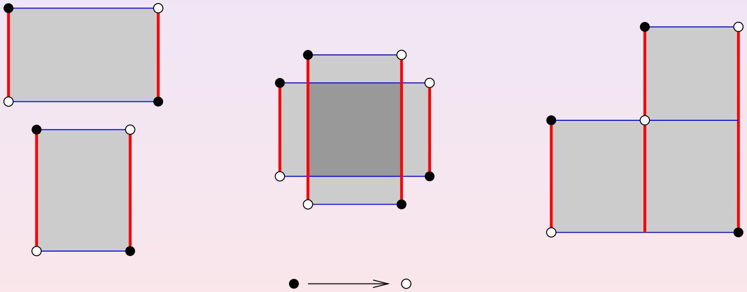
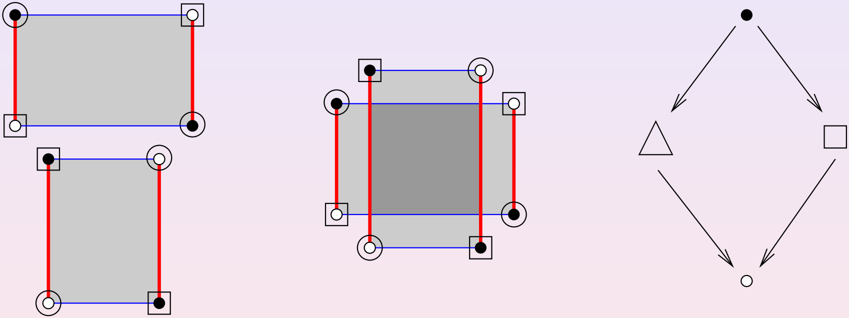
Why a chain complex? index two disks.

Let y be a generator appearing in $\partial^2 x$, i.e., there is a index two disk connecting x to y. It will looks like (let us just consider squares, for simplicity)

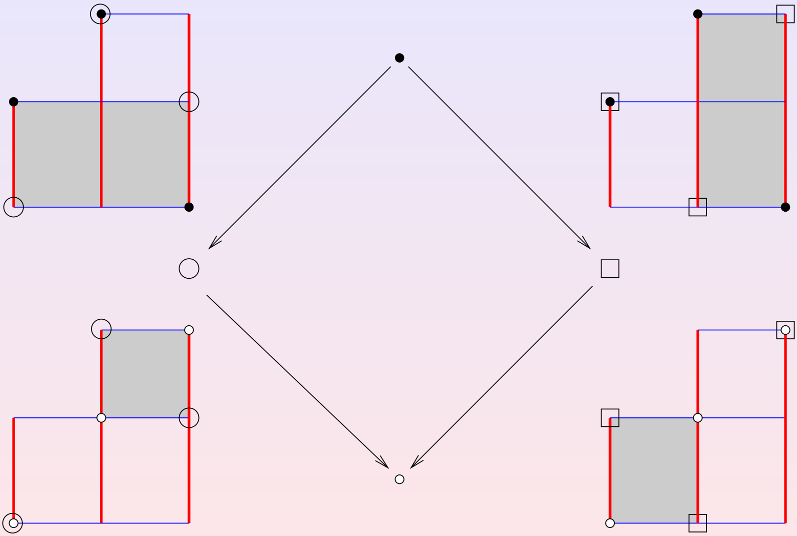


Why a chain complex? "Gromov compactness"



We see that the generator y (white dot) appears in $\partial^2 x$ in pairs. So $\partial^2 x = 0$ with $\mathbb{Z}_2$ coefficients.

Why a chain complex? "Gromov compactness", continued



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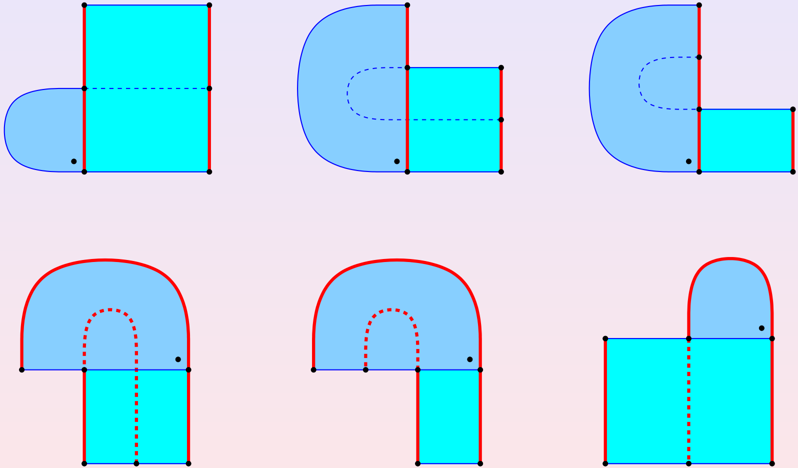
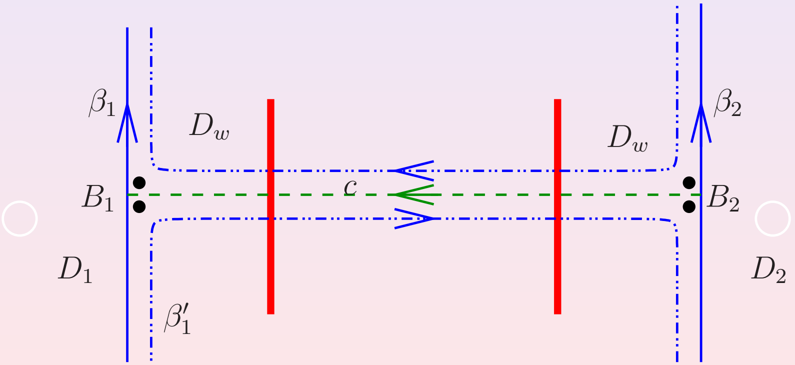


Proof of Equivalence, admissible handleslides

Proposition

A handleslide on a Heegaard diagram can be made admissible modulo admissible isotopies.



Proof of Equivalence, admissible isotopy

Proposition

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} w)$ be a pointed triple diagram. Suppose both $\mathcal{H}^1 = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ and $\mathcal{H}^2 = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, w)$ are nice diagrams and the beta and gamma curves are isotopic in the complement of w. Then $\mathcal{H}^1$ and $\mathcal{H}^2$ can be made identical after admissible moves and ambient isotopy of Σ.

Proof of equivalence

- Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are two nice diagrams for Y.
- They become equivalent after some admissible moves.
- Make the alpha curves isotopic in $\Sigma \setminus w$ after admisible handleslides.
- Make the two set of alpha curves identical.
- By admissible handleslides of beta curves, make beta and gamma curves isotopic in $\Sigma \setminus w$.
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Algebraic lemma

Let $(\mathcal{C}, \partial)$ be a graded chain complex generated by $G = \{g_1, \cdots, g_m\}$ and the differential ∂ is of degree -1. We write

$$\partial g_i = \sum_{j=1}^{m} a_i^j g_j.$$

Suppose $a_k^l = 1$. Let $\mathcal{C}'$ be the vector space generated by $G \setminus \{g_k, g_l\}$ with the same degree as in $\mathcal{C}$. Define

$$\partial'(g_i) := \sum_{j \neq k,l} (a_i^j + a_i^l a_k^j) g_j.$$

$$\Phi : \mathcal{C} \to \mathcal{C}', \quad \Phi(g_k) := 0, \quad \Phi(g_l) := \sum_{j \neq k,l} a_k^j g_j, \quad \Phi(g_j) := g_j \text{ (for } j \neq k, l)$$

$$\Psi : \mathcal{C}' \to \mathcal{C}, \quad \Psi(g_j) = g_j + a_j^l g_k \quad (j \neq k, l)$$

Lemma

$(\mathcal{C}', \partial')$ is a chain complex (that is, $\partial' \circ \partial' = 0$). Moreover, $(\mathcal{C}, \partial)$ and $(\mathcal{C}', \partial')$ are chain equivalent under the pair (Φ, Ψ).

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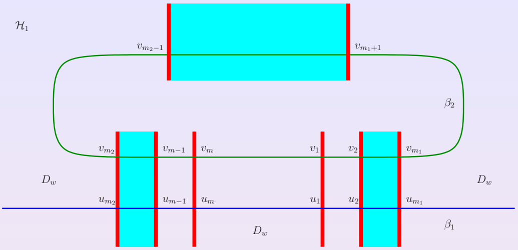
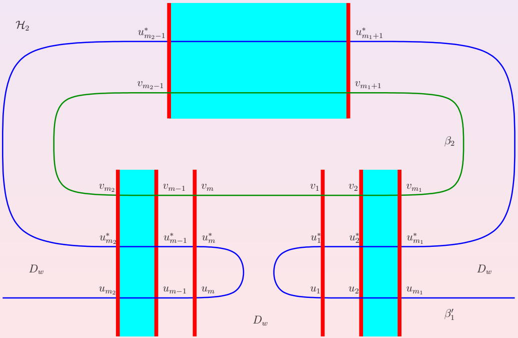
Proposition (Handleslide invariance)

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ are two nice diagrams which differ by an admissible handleslide. Then $\widehat{CC}(\mathcal{H}_1)$ and $\widehat{CC}(\mathcal{H}_2)$ are chain equivalent.



$\mathcal{H}_1$
v_{m_2-1}
v_{m_1+1}
β_2
v_{m_2}
v_{m-1}
v_m
v_1
v_2
v_{m_1}
D_w
D_w
u_{m_2}
u_{m-1}
u_m
u_1
u_2
u_{m_1}
D_w
β_1

(a)

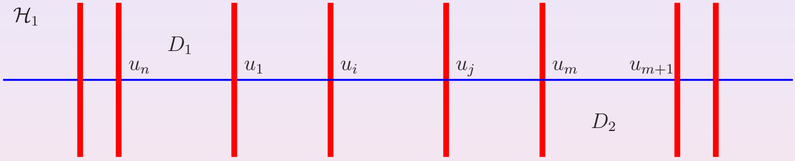


$\mathcal{H}_2$
$u^*_{m_2-1}$
$u^*_{m_1+1}$
v_{m_2-1}
v_{m_1+1}
β_2
v_{m_2}
v_{m-1}
v_m
v_1
v_2
v_{m_1}
$u^*_{m_2}$
u^*_{m-1}
u^*_m
u^*_1
u^*_2
$u^*_{m_1}$
D_w
D_w
u_{m_2}
u_{m-1}
u_m
u_1
u_2
u_{m_1}
D_w
β'_1

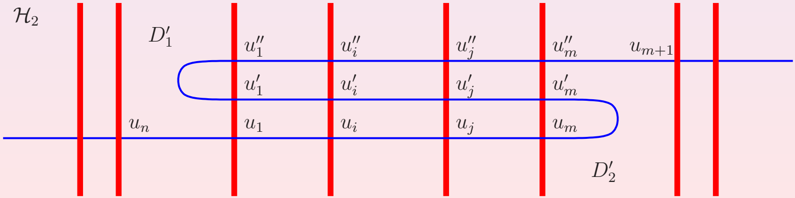
(b)

Proposition (isotopy invariance)

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ are two nice diagrams which differ by an admissible isotopy. Then $\widehat{CC}(\mathcal{H}_1)$ and $\widehat{CC}(\mathcal{H}_2)$ are chain equivalent.



(a)



(b)

☺

Thank you!

спасибо!

Я помню чудное мгновенье:

Передо мной явилась ты,

Как мимолетное виденье,

Как гений чистой красоты.