Hexagon relations and their cohomologies

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Outline

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Generalities

Quantum relations from set-theoretic relations and cohomologies

Pachner move 3–3 and set theoretic hexagon

Linear set theoretic hexagon

Quantum hexagon: cohomologies for $F = \mathbb{F}_2$

Newest development: cubic Lagrangian for a 4d TQFT

How 2-cocycles arise from Gaussian pentachoron weights

Recapitulation

What this report deals with:

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- Hexagon relations—algebraic realizations of four-dimensional Pachner moves, primarily, of the move 3–3.
- Topological quantum field theories (of sorts).

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- Quandles are good for knots and knotted surfaces, while *hexagon relations* are good for 4-manifolds. Undeformed hexagon relations are "set-theoretic".
- *Linear set-theoretic* relations may already contain in them one more "deformation" of a different character, determined by an arbitrary 2-cocycle.
- *Gaussian exponentials* of three types—*fermionic*, usual *bosonic*, and *discrete*—may also be used as building blocks for hexagon relations. These can be reduced, in a nontrivial way, to linear set-theoretic relations, at least in many cases.

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n-Gon relations

These are *pentagon*, *hexagon*, ...

Pentagon corresponds to 3-dimensional Pachner moves, hexagon to 4-dimensional, etc.

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Almost the same, but X may be infinite, like $X = \mathbb{R}$ or $\mathbb{C}$; the two subsets are typically *algebraic* in $X \times \cdots \times X$, and the relation states the equalness of their *Zariski closures*.

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Quantum (= tensor) relations

The relation states the equalness of two elements in a tensor product $V \otimes \cdots \otimes V$ of some copies of an $\mathbb{R}$- or $\mathbb{C}$-linear space V, (for simplicity, we assume $V^* = V$).

Outline

Quantum relations from set-theoretic relations

Without cohomologies

Linear space V has, by definition, set X as its *basis*.
Then, to each element

$$(x_1, \dots, x_n) \in X^{\times n} = \underbrace{X \times \cdots \times X}_{n}$$

corresponds basis vector

$$x_1 \otimes \cdots \otimes x_n \in V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n}.$$

The element in $V^{\otimes n}$ corresponding to a subset $S \subset X^{\times n}$ is

$$\sum_{(x_1,\dots,x_n)\in S} x_1 \otimes \cdots \otimes x_n.$$

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We now introduce scalar multipliers φ into the previous sum: the element in $V^{\otimes n}$ becomes now

$$\sum_{(x_1,\dots,x_n)\in S} \varphi(x_1,\dots,x_n)\, x_1 \otimes \cdots \otimes x_n. \qquad (*)$$

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Coboundaries

And some φ satisfy the cocycle condition in a trivial—uninteresting—way; such φ can be called *coboundaries*.

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pentachora ($=$ 4-simplices) 12345, 12346 and 12356, grouped thus around 2-face 123.

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pentachora 12456, 13456 and 23456, grouped around 2-face 456.

Both sides occupy the same place in a triangulation and have the same boundary,

consisting of the following nine tetrahedra:

1245	1246	1256
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Set theoretic hexagon relation

3-faces are colored

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For one pentachoron, let it be 12345, a subset

$$R_{12345} \subset X_{2345} \times X_{1345} \times X_{1245} \times X_{1235} \times X_{1234}$$

in the Cartesian product of copies of set X is given. Similarly, there are its copies R_u for five other pentachora u.

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Subset S_L corresponding to the l.h.s. of the relation:

$$(x_{1245}, x_{1246}, x_{1256}, x_{1345}, x_{1346}, x_{1356}, x_{2345}, x_{2346}, x_{2356}) \in S_L$$

provided there are such x_{1234}, x_{1235} and x_{1236}—corresponding to *inner* tetrahedra—that $(x_{2345}, x_{1345}, x_{1245}, x_{1235}, x_{1234}) \in R_{12345}$ and similarly for two other pentachora 12346 and 12356.

Then S_R in the similar way.

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And the relation is of course

$$S_L = S_R.$$

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Now let the set X of colors be a *two-dimensional linear space* over a field F. For each pentachoron u, let

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be a *five-dimensional linear subspace* in the ten-dimensional direct sum of copies of X.

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Constant and non-constant R_u

The simplest case is where R_u are, for all pentachora u, copies of one fixed R (in some natural sense). But when they are not, this brings about extremely interesting mathematical structures.

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The hexagon relation $S_L = S_R$

Generically (if no degeneracies), S_L and S_R are nine-dimensional linear subspaces of the same 18-dimensional space.

A simple example of constant $R_u = R$

For each tetrahedron t, we introduce a basis in X_t, and denote x_t, y_t the two coordinates of a vector $\mathbf{x}_t \in X_t$. Five-dimensional subspace $R_u \subset \bigoplus_{t\subset u} X_t$ can be given, for $u = 12345$, by the following five linear relations, and for any u similarly, changing $1, 2, 3, 4, 5$ to the vertices of u taken in their increasing order:

$$\begin{pmatrix} y_{2345} \\ y_{1345} \\ y_{1245} \\ y_{1235} \\ y_{1234} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{2345} \\ x_{1345} \\ x_{1245} \\ x_{1235} \\ x_{1234} \end{pmatrix}.$$

We will also write this, taking some notational liberty (concerning the letter R), as

$$\mathbf{y} = R\mathbf{x}.$$

Pentachoron weight and edge operators

The weight is not yet quantum! Here, it is a delta function. One more "not very quantum" weight is Gaussian, to be discussed a bit later.

Matrix R in the previous slide was obtained using the technique of *edge operators*. These annihilate the *pentachoron weight*, which is, in our case, the five-dimensional Dirac delta function or, in a discrete case, Kronecker delta symbol:

$$\mathcal{W} = \delta(\mathbf{y} - R\mathbf{x}).$$

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And an *edge operator* corresponding to an edge b is their linear combination involving only x's and y's on tetrahedra $t \supset b$.

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Edge operator for edge $b = 12$ $(= 21)$ has the form

$$\mathrm{e}_{12} = \alpha_{12}^{(1234)} x_{1234} + \beta_{12}^{(1234)} y_{1234} + \alpha_{12}^{(1235)} x_{1235} + \beta_{12}^{(1235)} y_{1235} + \alpha_{12}^{(1245)} x_{1245} + \beta_{12}^{(1245)} y_{1245} \,.$$

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Linear dependence in vertex 1:

$$\gamma_{12}\mathrm{e}_{12} + \gamma_{13}\mathrm{e}_{13} + \gamma_{14}\mathrm{e}_{14} + \gamma_{15}\mathrm{e}_{15} = 0.$$

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Remark: And for Gaussian pentachoron weights, edge operators are differential!

Some discussion will be below.

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The values ω_{ijk} of ω are related to the coefficients γ_{ij} in the previous slide:

$$\omega_{123} = \frac{\gamma_{12}}{\gamma_{21}}\frac{\gamma_{23}}{\gamma_{32}}\frac{\gamma_{31}}{\gamma_{13}}.$$

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- Δ^5 is a 5-simplex, and its boundary $\partial\Delta^5$ is the l.h.s. and r.h.s. of move 3–3 together. A coloring of (all 3-faces of) Δ^5 must induce a permitted coloring on each of its six pentachora.

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Now introduce $\mathbb{Z}_2$-valued cochains

These are arbitrary mappings from colorings to the group $\mathbb{Z}_2$. For cochains, we call the arrows δ_5 and δ_4, and they point to the *left*.

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Example of a nontrivial cocycle: an elegant cubic polynomial

$$x_{1345}x_{1235}x_{1234} + x_{1345}x_{1245}x_{1234} + x_{2345}x_{1245}x_{1234} + x_{2345}x_{1245}x_{1235} + x_{2345}x_{1345}x_{1235}$$

for pentachoron 12345, and the same with $1 \mapsto i, \ldots, 5 \mapsto m$ for other five pentachora $ijklm, \quad i < j < k < l < m$.

We of course identify $\mathbb{Z}_2$ with the additive group of $\mathbb{F}_2$.

Outline

Newest development: cubic Lagrangian for a four-dimensional topological quantum field theory

And over a field of characteristic 2

Exotic “discrete Lagrangian density” is proposed for topological quantum field theories on piecewise linear four-manifolds. It has the form of a cubic polynomial over a finite field of characteristic two.

Field

We choose a finite field $F = \mathbb{F}_{2^n}$ of characteristic two. For instance, field $\mathbb{F}_2$ of two elements, or $\mathbb{F}_4$ of four elements, etc.

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All vertices of the triangulation of M are numbered from 1 through their total number N_0.

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Vertices and pentachora

All vertices of the triangulation of M are numbered from 1 through their total number N_0.

Below we will do some constructions for each pentachoron. We will demonstrate them on the example of pentachoron 12345, keeping in mind that the change

$$1 \mapsto i, \quad \dots, \quad 5 \mapsto m.$$

should be done for an arbitrary pentachoron $ijklm$, where the vertices go in the *increasing order:* $i < j < k < l < m$.

Variables

We put a variable x_t on each 3-face t of the triangulation, taking values in the mentioned field: $x_t \in F = \mathbb{F}_{2^n}$.

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More variables

For each pentachoron u, we define quantities $y_t^{(u)} \in F$ living on all its 3-faces t. For pentachoron 12345, they are given be

$$\begin{pmatrix} y_{2345} \\ y_{1345} \\ y_{1245} \\ y_{1235} \\ y_{1234} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{2345} \\ x_{1345} \\ x_{1245} \\ x_{1235} \\ x_{1234} \end{pmatrix}.$$

Here, of course, $y_{2345} = y_{2345}^{(12345)}$, etc.

Restrictions on variables x_t

All sets of variables x_t comprise an F-linear space, namely, F^{N_3}, where N_3 is the total number of tetrahedra. We will, however, allow our variables only to run over its *subspace* $L \subset F^{N_3}$ determined by the following restrictions.

According to the previous slide, two y_t appear on each tetrahedron t, because t is a common 3-face for two neighboring pentachora, denote them u_1 and u_2. The subspace L is singled out by the requirement that these two y_t *coincide for every* t:

$$y_t^{(u_1)} = y_t^{(u_2)}.$$

Lagrangian density

For each tuple of variables x_t, *belonging to the subspace* L, we calculate the value of the following polynomial. First, we calculate for each pentachoron u what can be called "discrete Lagrangian density" A_u. For pentachoron 12345, it is

$$A_{12345} = x_{1345}x_{1235}x_{1234} + x_{1345}x_{1245}x_{1234} + x_{2345}x_{1245}x_{1234} + x_{2345}x_{1245}x_{1235} + x_{2345}x_{1345}x_{1235}.$$

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Action

Then these all are summed up, and we obtain our “action”:

$$A = \sum_u A_u.$$

PL manifold invariants

First, just a 'corrected' number of colorings

'Rough' invariant, not requiring our cocycle/action:

$$I_{\text{rough}}(M) = \dim L - \frac{1}{2} N_4 - 2N_0.$$

Here N_4 and N_0 are the numbers of pentachora and vertices in the triangulation. Note that

$$\dim L = \log_{|F|}(\text{total number of permitted colorings}).$$

So, this invariant is a clear analogue of quandle invariants not using cohomologies.

Some calculation results:

$$I_{\text{rough}}(S^4) = -6, \qquad I_{\text{rough}}(S^2 \times S^2) = -10, \qquad I_{\text{rough}}(T^4) = 6.$$

Remark: this holds for *any* finite field of characteristic two.

PL manifold invariants

Now, our action comes into play

Action A also takes values in $F = \mathbb{F}_{2^n}$. For any $v \in F$, the *'probability of value v for the action'* is a manifold invariant:

$$P_v(M, F) = \frac{\#v}{\text{(total number of permitted colorings)}}.$$

Here $\#v$—the *multiplicity* of v—is the number of times A takes value v when variables x_t run through subspace L.

Notations: Below we also denote $P_v(M, F) = \mathrm{P}(v)$.

Remark: this can be also reformulated in terms of 2^n state sum invariants, each using its group homomorphism

$$\mathbb{F}_{2^n} \to \mathbb{C}^*$$

as an 'exponential function', with values only 1 and -1, of course.

Calculation of invariants: first results

This turns out not very easy and requires computer resources or/and further theory development. And the results below are *preliminary!*

Calculations are being made right these days by my young colleague *Nurlan Sadykov*, using his specialized GAP package *PLGAP* for calculations with PL manifolds.

For $F = \mathbb{F}_4$:

M	$\mathrm{P}(0)$	$\mathrm{P}(1)$	P(any other value)
S^4	1	0	0
$S^2 \times S^2$	$5/8$	$3/8$	0
T^4	$65/128$	$63/128$	0

For $F = \mathbb{F}_8$:

M	$\mathrm{P}(0)$	P(any other value)
S^4	1	0
$S^2 \times S^2$	$11/32$	$3/32$
T^4	$71/512$	$63/512$

Outline

How 2-cocycles arise from Gaussian pentachoron weights

For instance, for the "two-boson" case. Here is a 10×10 symmetric matrix whose *pairs* of rows and columns correspond to tetrahedra—3-faces of pentachoron 12345.

Gaussian weight is of course exp(quadratic form).

$$\left(\begin{array}{cc|cc|cc|cc|cc}
f_{11} & g_{11} & f_{12} & g_{12} & f_{13} & g_{13} & f_{14} & g_{14} & f_{15} & g_{15} \\
g_{11} & r_{11} & h_{12} & r_{12} & h_{13} & r_{13} & h_{14} & r_{14} & h_{15} & r_{15} \\
\hline
f_{12} & h_{12} & f_{22} & g_{22} & f_{23} & g_{23} & f_{24} & g_{24} & f_{25} & g_{25} \\
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Where is a 2-cocycle hidden?

How 2-cocycles arise from Gaussian pentachoron weights

Here is the explanation. Recall that, for instance, $W = \exp(-x^2/2)$ satisfies the equation $\left(\dfrac{d}{dx} + x\right) W = 0$, and there are similar annihilating differential operators for multi-variable Gaussian exponentials as well.

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2-cocycle from coefficients of these dependencies

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And this is THE END, thank you.