Hexagon relations and their cohomologies

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Outline

Generalities

Quantum relations from set-theoretic relations and cohomologies

Pachner move 3–3 and set theoretic hexagon

Linear set theoretic hexagon

Quantum hexagon: cohomologies for $F = \mathbb{F}_2$

Newest development: cubic Lagrangian for a 4d TQFT

How 2-cocycles arise from Gaussian pentachoron weights

Recapitulation
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Recapitulation
What this report deals with:

I Piecewise linear (PL) four-dimensional manifold $M$.
I Hexagon relations/algebraic realizations of four-dimensional Pachner moves, primarily, of the move $3\{3$.
I Topological quantum field theories (of sorts).
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What \textit{quantum} means in four dimensions
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- A realistic (＝feasible) deformation in four dimensions consists in using cohomologies of algebraic structures, e.g., quandles, instead of using them without cohomologies. This means *deforming* a trivial cohomology class into a nontrivial one.
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- A realistic (= feasible) deformation in four dimensions consists in using cohomologies of algebraic structures, e.g., quandles, instead of using them without cohomologies. This means *deforming* a trivial cohomology class into a nontrivial one.
- Quandles are good for knots and knotted surfaces, while *hexagon relations* are good for 4-manifolds. Undeformed hexagon relations are “set-theoretic”.
- Linear set-theoretic relations may already contain in them one more *deformation* of a different character, determined by an arbitrary 2-cocycle.
- Gaussian exponentials of three types—fermionic, usual bosonic, and discrete—may also be used as building blocks for hexagon relations. These can be reduced, in a nontrivial way, to linear set-theoretic relations, at least in many cases.
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$n$-Simplex relations and $n$-gon relations

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These are *Yang–Baxter, tetrahedron, 4-simplex, ...*

They come mainly from mathematical physics. Some cases of Yang–Baxter and its simplified version, quandles, correspond also very naturally to Reidemeister/Roseman moves in the theory of knots/knotted surfaces.

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**n-Gon relations**

These are *pentagon, hexagon, ...*  
Pentagon corresponds to 3-dimensional Pachner moves, hexagon to 4-dimensional, etc.
Typical versions of $n$-simplex and $n$-gon relations
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Set-theoretic relations
There is a set $X$ of “colors”, and the relation states the equalness of two subsets in a direct product $X \times \cdots \times X$ of some copies of $X$. 
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**Functional relations**
Almost the same, but $X$ may be infinite, like $X = \mathbb{R}$ or $\mathbb{C}$; the two subsets are typically *algebraic* in $X \times \cdots \times X$, and the relation states the equalness of their *Zariski closures*. 
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Almost the same, but \( X \) may be infinite, like \( X = \mathbb{R} \) or \( \mathbb{C} \); the two subsets are typically algebraic in \( X \times \cdots \times X \), and the relation states the equalness of their Zariski closures.

Quantum (= tensor) relations
The relation states the equalness of two elements in a tensor product \( V \otimes \cdots \otimes V \) of some copies of an \( \mathbb{R} \)- or \( \mathbb{C} \)-linear space \( V \), (for simplicity, we assume \( V^* = V \)).
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Recapitulation
Quantum relations from set-theoretic relations

Without cohomologies

Linear space $V$ has, by definition, set $X$ as its basis. Then, to each element 

$$(x_1, \ldots, x_n) \in X \times_n = \underbrace{X \times \cdots \times X}_{n}$$

corresponds basis vector 

$$x_1 \otimes \cdots \otimes x_n \in V \otimes_n = \underbrace{V \otimes \cdots \otimes V}_{n}.$$ 

The element in $V \otimes_n$ corresponding to a subset $S \subset X \times_n$ is 

$$\sum_{(x_1, \ldots, x_n) \in S} x_1 \otimes \cdots \otimes x_n.$$
Introducing cohomologies

Cohomology group is the factor cocycles/coboundaries

We now introduce scalar multipliers into the previous sum: the element in $V^n$ becomes now $X(x_1;\ldots;x_n)\in S'(x_1;\ldots;x_n)x_1^n$.

Cocycles

Recall that two expressions of type $(\ldots)$, corresponding to the l.h.s. and r.h.s., must be equal. This can usually be interpreted as a cocycle condition.

Coboundaries

And some $\epsilon$'s satisfy the cocycle condition in a trivial|uninteresting|way; such $\epsilon$'s can be called coboundaries.
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We now introduce scalar multipliers $\varphi$ into the previous sum: the element in $V^\otimes n$ becomes now

$$\sum_{(x_1,\ldots,x_n) \in S} \varphi(x_1,\ldots,x_n) x_1 \otimes \cdots \otimes x_n. \quad (*)$$
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Left-hand side (that is, initial state):
pentachora ( = 4-simplices) 12345, 12346 and 12356, grouped thus around 2-face 123.
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Right-hand side (final state):
pentachora 12456, 13456 and 23456, grouped around 2-face 456.
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Right-hand side (final state):
pentachora 12456, 13456 and 23456, grouped around 2-face 456.

Both sides occupy the same place in a triangulation and have the same boundary,
consisting of the following nine tetrahedra:

\[
\begin{array}{ccc}
1245 & 1246 & 1256 \\
1345 & 1346 & 1356 \\
2345 & 2346 & 2356 \\
\end{array}
\]
Set theoretic hexagon relation

3-faces are colored
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For one pentachoron, let it be 12345, a subset

\[ R_{12345} \subseteq X_{2345} \times X_{1345} \times X_{1245} \times X_{1235} \times X_{1234} \]

in the Cartesian product of copies of set \( X \) is given. Similarly, there are its copies \( R_u \) for five other pentachora \( u \).
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Subset \( S_L \) corresponding to the l.h.s. of the relation:

\[ (x_{1245}, x_{1246}, x_{1256}, x_{1345}, x_{1346}, x_{1356}, x_{2345}, x_{2346}, x_{2356}) \in S_L \]

provided there are such \( x_{1234} \), \( x_{1235} \) and \( x_{1236} \)—corresponding to \textit{inner} tetrahedra—that \( (x_{2345}, x_{1345}, x_{1245}, x_{1235}, x_{1234}) \in R_{12345} \) and similarly for two other pentachora 12346 and 12356.

Then \( S_R \) in the similar way.
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And the relation is of course
\[ S_L = S_R. \]
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Recapitulation
Linear set theoretic hexagon

Now let the set $X$ of colors be a two-dimensional linear space over a field $F$. For each pentachoron $u$, let $R_u$ be a five-dimensional linear subspace in the ten-dimensional direct sum of copies of $X$.

Constant and non-constant $R_u$ The simplest case is where $R_u$ are, for all pentachora $u$, copies of one fixed $R$ (in some natural sense). But when they are not, this brings about extremely interesting mathematical structures.

The hexagon relation $S_L = S_R$ Generically (if no degeneracies), $S_L$ and $S_R$ are nine-dimensional linear subspaces of the same 18-dimensional space.
Linear set theoretic hexagon

Linear set of colors
Now let the set $X$ of colors be a *two-dimensional linear space* over a field $F$. For each pentachoron $u$, let

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Generically (if no degeneracies), $S_L$ and $S_R$ are nine-dimensional linear subspaces of the same 18-dimensional space.
A simple example of constant $R_u = R$

For each tetrahedron $t$, we introduce a basis in $X_t$, and denote $x_t, y_t$ the two coordinates of a vector $x_t \in X_t$. Five-dimensional subspace $R_u \subset \bigoplus_{t \subseteq u} X_t$ can be given, for $u = 12345$, by the following five linear relations, and for any $u$ similarly, changing $1, 2, 3, 4, 5$ to the vertices of $u$ taken in their increasing order:

$$
\begin{pmatrix}
  y_{2345} \\
  y_{1345} \\
  y_{1245} \\
  y_{1235} \\
  y_{1234}
\end{pmatrix} =
\begin{pmatrix}
  0 & 0 & 1 & -1 & 0 \\
  0 & 1 & 0 & -1 & 1 \\
  1 & 0 & 0 & 0 & 1 \\
  1 & -1 & 0 & 1 & 0 \\
  0 & -1 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  x_{2345} \\
  x_{1345} \\
  x_{1245} \\
  x_{1235} \\
  x_{1234}
\end{pmatrix}.
$$

We will also write this, taking some notational liberty (concerning the letter $R$), as

$$
y = Rx.
$$
Pentachoron weight and edge operators

The weight is not yet quantum! Here, it is a delta function. One more “not very quantum” weight is Gaussian, to be discussed a bit later.

Matrix $R$ in the previous slide was obtained using the technique of edge operators. These annihilate the pentachoron weight, which is, in our case, the five-dimensional Dirac delta function or, in a discrete case, Kronecker delta symbol:

$$\mathcal{W} = \delta(y - Rx).$$
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And an edge operator corresponding to an edge $b$ is their linear combination involving only $x$’s and $y$’s on tetrahedra $t \supset b$. 
An example edge operator and an example linear dependence between them.
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Edge operator for edge $b = 12 (= 21)$ has the form

$$e_{12} = \alpha_{12}^{(1234)} x_{1234} + \beta_{12}^{(1234)} y_{1234} + \alpha_{12}^{(1235)} x_{1235} + \beta_{12}^{(1235)} y_{1235} + \alpha_{12}^{(1245)} x_{1245} + \beta_{12}^{(1245)} y_{1245}.$$
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$$+ \alpha_{12}^{(1245)} x_{1245} + \beta_{12}^{(1245)} y_{1245}.$$ 

Linear dependence in vertex 1:

$$\gamma_{12} e_{12} + \gamma_{13} e_{13} + \gamma_{14} e_{14} + \gamma_{15} e_{15} = 0.$$ 

For a constant $R$, all coefficients $\gamma_{ij}$ can be taken equal to unity.
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Remark: And for Gaussian pentachoron weights, edge operators are differential!

Some discussion will be below.
A few words about non-constant $R_u$

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The weight is still not quantum!

In the case $R_u \neq \text{const}$, it has a nonlinear parameterization in terms of a multiplicative 2-cocycle $\omega$ taking values in $F^*$ (recall that $F$ is a field). That is, for instance,

$$\frac{\omega_{123} \omega_{134}}{\omega_{124} \omega_{234}} = 1, \quad \omega_{123} = \omega_{213}^{-1} = \ldots.$$
A few words about non-constant $R_u$

The weight is still not quantum!

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\frac{\omega_{123} \omega_{134}}{\omega_{124} \omega_{234}} = 1, \quad \omega_{123} = \omega_{213}^{-1} = \ldots.
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The values $\omega_{ijk}$ of $\omega$ are related to the coefficients $\gamma_{ij}$ in the previous slide:

\[
\omega_{123} = \frac{\gamma_{12}}{\gamma_{21}} \frac{\gamma_{23}}{\gamma_{32}} \frac{\gamma_{31}}{\gamma_{13}}.
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And for the mentioned constant $R$, *not* involving any cocycle
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Mappings of permitted colorings:

(colorings of $\Delta^5$) $\xrightarrow{f_5}$ (pentachora colorings) $\xrightarrow{f_4}$ (3-faces colorings)
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- $\Delta^5$ is a 5-simplex, and its boundary $\partial \Delta^5$ is the l.h.s. and r.h.s. of move 3–3 together. A coloring of (all 3-faces of) $\Delta^5$ must induce a permitted coloring on each of its six pentachora.
Quantum hexagon: cohomologies for $F = \mathbb{F}_2$

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Mappings of permitted colorings:
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Now introduce $\mathbb{Z}_2$-valued cochains

These are arbitrary mappings from colorings to the group $\mathbb{Z}_2$. For cochains, we call the arrows $\delta_5$ and $\delta_4$, and they point to the left.
Quantum hexagon: cohomologies for $F = \mathbb{F}_2$

Calculation results, including a \textit{cubic} polynomial that can be called the Lagrangian of a TQFT.
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Example of a nontrivial cocycle: an elegant cubic polynomial

$$x_{1345}x_{1235}x_{1234} + x_{1345}x_{1245}x_{1234}$$
$$+ x_{2345}x_{1245}x_{1234} + x_{2345}x_{1245}x_{1235} + x_{2345}x_{1345}x_{1235}$$

for pentachoron 12345, and the same with $1 \mapsto i, \ldots, 5 \mapsto m$ for other five pentachora $ijklm$, $i < j < k < l < m$.

We of course identify $\mathbb{Z}_2$ with the additive group of $\mathbb{F}_2$. 
Outline

Generalities

Quantum relations from set-theoretic relations and cohomologies

Pachner move 3–3 and set theoretic hexagon

Linear set theoretic hexagon

Quantum hexagon: cohomologies for $F = \mathbb{F}_2$

Newest development: cubic Lagrangian for a 4d TQFT

How 2-cocycles arise from Gaussian pentachoron weights

Recapitulation
Newest development: cubic Lagrangian for a four-dimensional topological quantum field theory
And over a field of characteristic 2

Exotic “discrete Lagrangian density” is proposed for topological quantum field theories on piecewise linear four-manifolds. It has the form of a cubic polynomial over a finite field of characteristic two.
Field
We choose a finite field $F = \mathbb{F}_{2^n}$ of characteristic two. For instance, field $\mathbb{F}_2$ of two elements, or $\mathbb{F}_4$ of four elements, etc.
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Vertices and pentachora
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Vertices and pentachora
All vertices of the triangulation of $M$ are numbered from 1 through their total number $N_0$.
Below we will do some constructions for each pentachoron. We will demonstrate them on the example of pentachoron 12345, keeping in mind that the change

$$1 \mapsto i, \ldots, 5 \mapsto m.$$ 

should be done for an arbitrary pentachoron $ijklm$, where the vertices go in the increasing order: $i < j < k < l < m$. 
Variables

We put a variable $x_t$ on each 3-face $t$ of the triangulation, taking values in the mentioned field: $x_t \in F = \mathbb{F}_{2^n}$. 
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More variables
For each pentachoron $u$, we define quantities $y_t^{(u)} \in F$ living on all its 3-faces $t$. For pentachoron $12345$, they are given be

$$
\begin{pmatrix}
    y_{2345} \\
    y_{1345} \\
    y_{1245} \\
    y_{1235} \\
    y_{1234}
\end{pmatrix} =
\begin{pmatrix}
    0 & 0 & 1 & 1 & 0 \\
    0 & 1 & 0 & 1 & 1 \\
    1 & 0 & 0 & 0 & 1 \\
    1 & 1 & 0 & 1 & 0 \\
    0 & 1 & 1 & 0 & 0
\end{pmatrix} 
\begin{pmatrix}
    x_{2345} \\
    x_{1345} \\
    x_{1245} \\
    x_{1235} \\
    x_{1234}
\end{pmatrix}.
$$

Here, of course, $y_{2345} = y_{2345}^{(12345)}$, etc.
Restrictions on variables $x_t$

All sets of variables $x_t$ comprise an $F$-linear space, namely, $F^{N_3}$, where $N_3$ is the total number of tetrahedra. We will, however, allow our variables only to run over its subspace $L \subset F^{N_3}$ determined by the following restrictions.

According to the previous slide, two $y_t$ appear on each tetrahedron $t$, because $t$ is a common 3-face for two neighboring pentachora, denote them $u_1$ and $u_2$. The subspace $L$ is singled out by the requirement that these two $y_t$ coincide for every $t$:

$$y_t^{(u_1)} = y_t^{(u_2)}.$$
Lagrangian density

For each tuple of variables $x_t$, belonging to the subspace $L$, we calculate the value of the following polynomial. First, we calculate for each pentachoron $u$ what can be called “discrete Lagrangian density” $A_u$. For pentachoron 12345, it is

$$A_{12345} = x_{1345}x_{1235}x_{1234} + x_{1345}x_{1245}x_{1234} + x_{2345}x_{1245}x_{1234}$$
$$+ x_{2345}x_{1245}x_{1235} + x_{2345}x_{1345}x_{1235}.$$
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$$+ x_{2345}x_{1245}x_{1235} + x_{2345}x_{1345}x_{1235}.$$ 

Action

Then these all are summed up, and we obtain our “action”:

$$A = \sum_u A_u.$$
PL manifold invariants

First, just a ‘corrected’ number of colorings

‘Rough’ invariant, not requiring our cocycle/action:

\[ I_{\text{rough}}(M) = \dim L - \frac{1}{2} N_4 - 2 N_0. \]

Here \( N_4 \) and \( N_0 \) are the numbers of pentachora and vertices in the triangulation. Note that

\[ \dim L = \log|F|(\text{total number of permitted colorings}). \]

So, this invariant is a clear analogue of quandle invariants not using cohomologies.

Some calculation results:

\[ I_{\text{rough}}(S^4) = -6, \quad I_{\text{rough}}(S^2 \times S^2) = -10, \quad I_{\text{rough}}(T^4) = 6. \]

Remark: this holds for any finite field of characteristic two.
Action $A$ also takes values in $F = \mathbb{F}_{2^n}$. For any $v \in F$, the ‘probability of value $v$ for the action’ is a manifold invariant:

$$P_v(M, F) = \frac{\#v}{\text{total number of permitted colorings}}.$$ 

Here $\#v$—the multiplicity of $v$—is the number of times $A$ takes value $v$ when variables $x_t$ run through subspace $L$.

**Notations:** Below we also denote $P_v(M, F) = P(v)$.

**Remark:** this can be also reformulated in terms of $2^n$ state sum invariants, each using its group homomorphism

$$\mathbb{F}_{2^n} \to \mathbb{C}^*$$

as an ‘exponential function’, with values only 1 and $-1$, of course.
Calculation of invariants: first results

This turns out not very easy and requires computer resources or/and further theory development. And the results below are preliminary!

Calculations are being made right these days by my young colleague Nurlan Sadykov, using his specialized GAP package PLGAP for calculations with PL manifolds.

### For $F = \mathbb{F}_4$:

<table>
<thead>
<tr>
<th>$M$</th>
<th>$P(0)$</th>
<th>$P(1)$</th>
<th>$P($any other value$)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^4$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$S^2 \times S^2$</td>
<td>5/8</td>
<td>3/8</td>
<td>0</td>
</tr>
<tr>
<td>$T^4$</td>
<td>65/128</td>
<td>63/128</td>
<td>0</td>
</tr>
</tbody>
</table>

### For $F = \mathbb{F}_8$:

<table>
<thead>
<tr>
<th>$M$</th>
<th>$P(0)$</th>
<th>$P($any other value$)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^4$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$S^2 \times S^2$</td>
<td>11/32</td>
<td>3/32</td>
</tr>
<tr>
<td>$T^4$</td>
<td>71/512</td>
<td>63/512</td>
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How 2-cocycles arise from Gaussian pentachoron weights

Recapitulation
How 2-cocycles arise from Gaussian pentachoron weights

For instance, for the “two-boson” case. Here is a $10 \times 10$ symmetric matrix whose \textit{pairs} of rows and columns correspond to tetrahedra—3-faces of pentachoron 12345.

Gaussian weight is of course $\exp(\text{quadratic form})$.

\[
\begin{pmatrix}
  f_{11} & g_{11} & f_{12} & g_{12} & f_{13} & g_{13} & f_{14} & g_{14} & f_{15} & g_{15} \\
  g_{11} & r_{11} & h_{12} & r_{12} & h_{13} & r_{13} & h_{14} & r_{14} & h_{15} & r_{15} \\
  f_{12} & h_{12} & f_{22} & g_{22} & f_{23} & g_{23} & f_{24} & g_{24} & f_{25} & g_{25} \\
  g_{12} & r_{12} & g_{22} & r_{22} & h_{23} & r_{23} & h_{24} & r_{24} & h_{25} & r_{25} \\
  f_{13} & h_{13} & f_{23} & h_{23} & f_{33} & g_{33} & f_{34} & g_{34} & f_{35} & g_{35} \\
  g_{13} & r_{13} & g_{23} & r_{23} & g_{33} & r_{33} & h_{34} & r_{34} & h_{35} & r_{35} \\
  f_{14} & h_{14} & f_{24} & h_{24} & f_{34} & h_{34} & f_{44} & g_{44} & f_{45} & g_{45} \\
  g_{14} & r_{14} & g_{24} & r_{24} & g_{34} & r_{34} & g_{44} & r_{44} & h_{45} & r_{45} \\
  f_{15} & h_{15} & f_{25} & h_{25} & f_{35} & h_{35} & f_{45} & h_{45} & f_{55} & g_{55} \\
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  g_{14} & r_{14} & g_{24} & r_{24} & g_{34} & r_{34} & g_{44} & r_{44} & g_{45} & r_{45} \\
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\end{pmatrix}
$$

Where is a 2-cocycle hidden?
How 2-cocycles arise from Gaussian pentachoron weights

Here is the explanation. Recall that, for instance, $W = \exp(-x^2/2)$ satisfies the equation $(\frac{d}{dx} + x)W = 0$, and there are similar annihilating differential operators for multi-variable Gaussian exponentials as well.
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Gaussian weight
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\downarrow \\
\text{Linear space of operators annihilating the Gaussian weight} \\
\downarrow
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\[ \downarrow \]
\[ \text{Linear space of operators annihilating the Gaussian weight} \]
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\]
\[
\text{Linear space of operators annihilating the Gaussian weight} \downarrow
\]
\[
\text{Edge operators} \downarrow
\]
\[
\text{Linear dependencies between edge operators in each vertex} \downarrow
\]
\[
\text{2-cocycle from coefficients of these dependencies}
\]
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Now some recapitulation
Now some recapitulation

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- Quantum hexagon relations contain further deformations due to cohomologies.
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And this is **THE END**, thank you.