2-Colored diagrams of knots: moves, realizability and invariants

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Moscow, 05 July 2017

Outline

Knot definition and knot recognition problem

Definition

A *knot* is the image of a smooth embedding of the standard circle S^1 in a three-dimensional sphere S^3 (or in $\mathbb{R}^3$). If we fix an orientation of the circle S^1, then we have an *oriented knot*)

Definition

Two knots are called *isotopic*, if one of them can be transformed to the other by a diffeomorphism (preserving the orientation) of the ambient space onto itself (such a diffeomorphism is called an *isotopy*).

Planar knot diagram

Definition

A *knot diagram* is a projection in general position of a knot on a plane $\mathbb{R}^2 \subset \mathbb{R}^3$ with additional structure of crossings.

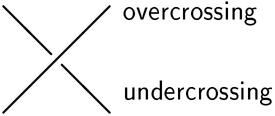


Figure: The local structure of a crossing

Reidemeister moves

Theorem (Reidemeister)

Two knot diagrams represent isotopic knots if and only if they are connected by a finite sequence of Reidemeister moves.

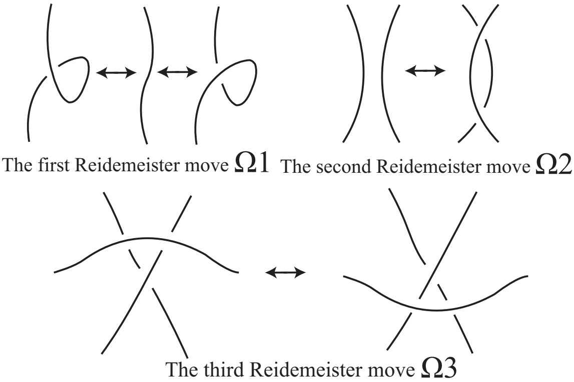
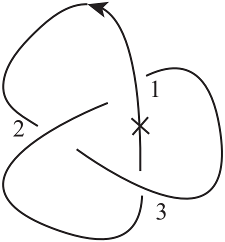


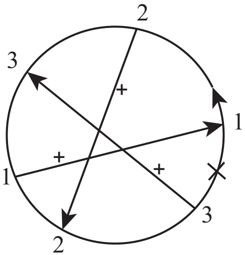
Figure: Reidemeister moves $\Omega 1$, $\Omega 2$, $\Omega 3$

Gauss diagrams

Definition

By the *Gauss diagram* corresponding to a (oriented) knot diagram we mean a diagram consisting of an oriented circle (with a fixed point, not a preimage of a crossing) on which the preimages of the overcrossings and the undercrossings are connected by arrows directed from the preimages of the undercrossings to the preimages of the overcrossings. Each arrow is endowed with a sign coinciding with the sign of the corresponding crossing, i.e. it equals 1 if we have  and -1 for .





Moves on Gauss diagrams

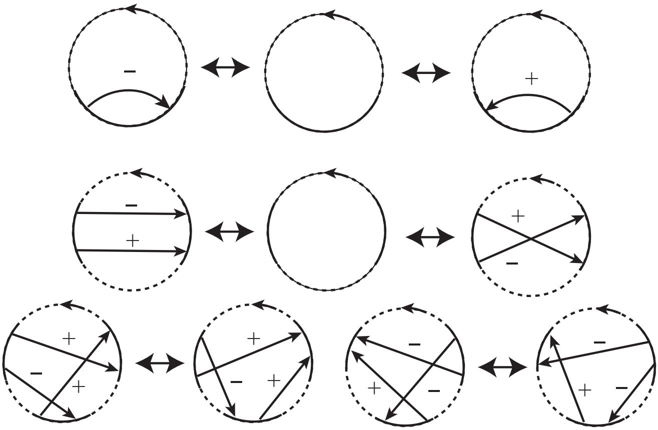


Figure: Moves on Gauss diagrams

2-colored diagrams

Let K be a knot diagram with two fixed points not coinciding with crossings. Those two points divide K into two curves γ_1 and γ_2.

Definition

K is called 2-*colored* if the curves γ_i $(i = 1, 2)$ have no self-crossings.

Construction of a 2-colored diagram

Theorem

Each knot has a 2-colored diagram.

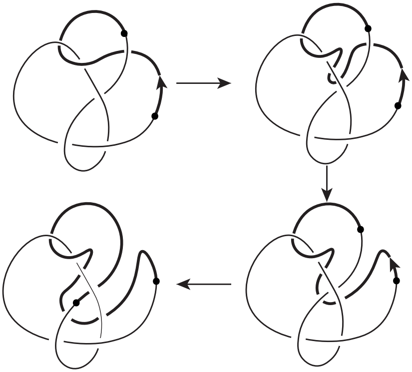


Figure: Constructing of a 2-colored diagram

Construction of a 2-colored diagram

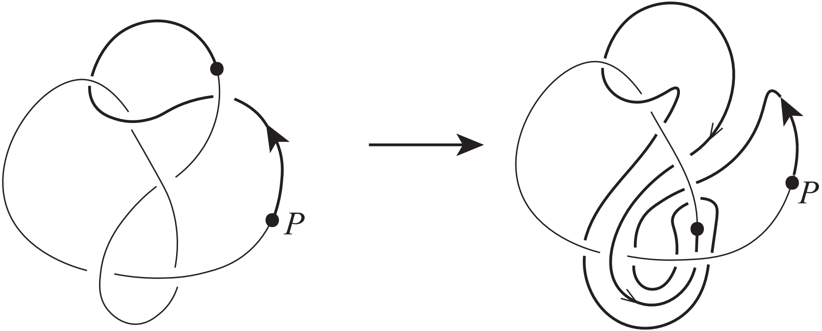


Figure: A 2-colored diagram

Construction of a 2-colored diagram

Remark

Note that a result of the algorithm from the theorem depends on the starting point and the direction of the movement.

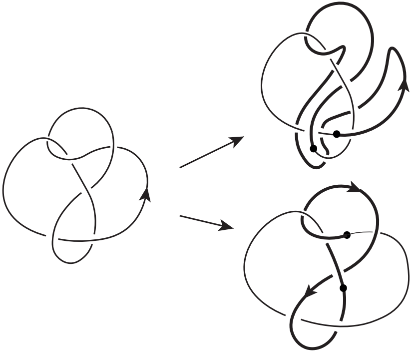


Figure: Different 2-colored diagrams

Moves on 2-colored diagrams: regular moves

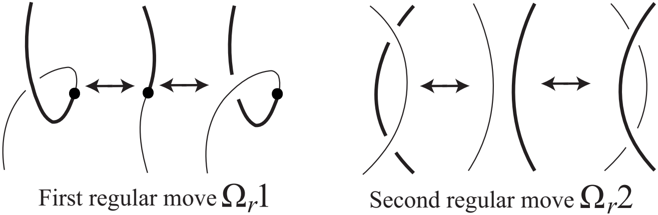


Figure: Moves Ω_r1, Ω_r2 on 2-colored diagrams

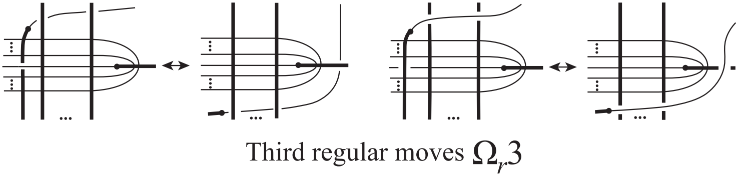


Figure: Moves Ω_r3 on 2-colored diagrams

Regular equivalence

Definition

We call two 2-colored diagrams *regular equivalent* if one of them can be obtained from the other by a finite chain of regular moves and planar isotopies.

Theorem

Two 2-colored diagrams are the diagrams of isotopic knots if and only if they are regular equivalent.

Lemma 1

Lemma

Let K be any knot diagram and P, Q be two points on K, different from crossings. Then two 2-colored diagrams of the knot obtained by applying the algorithm of constructing 2-colored diagram to P and Q of K with a fixed orientation are regular equivalent.

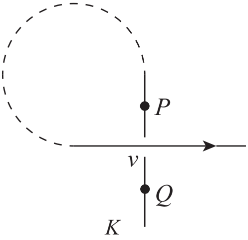


Figure: The knot diagram K

Lemma 1

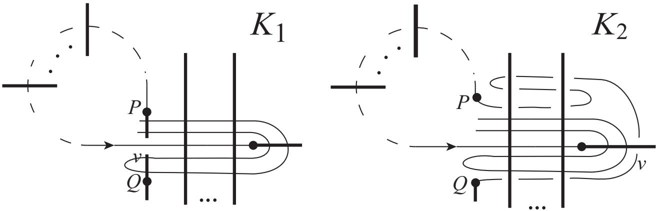


Figure: Two 2-colored diagrams K_1 and K_2

Lemma 1

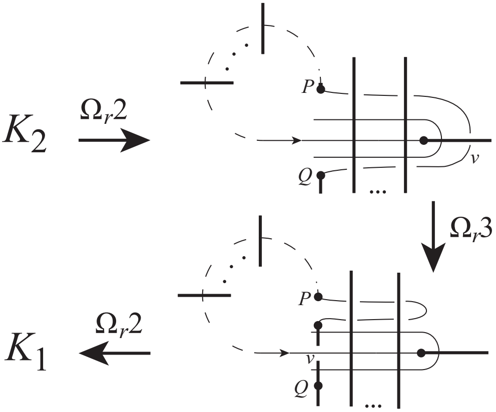


Figure: Regular moves from K_2 to K_1

Lemma 2

Lemma

Let a diagram K_1 be obtained from a diagram K_2 by applying a first Reidemeister move. Then two 2-colored diagrams of the knot obtained by applying the algorithm of obtaining 2-colored diagram to K_1 and K_2 with the same orientations are regular equivalent.

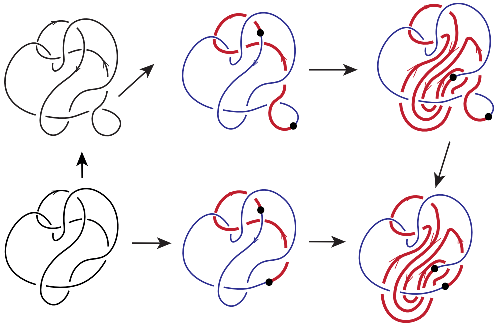


Figure: Example

Lemma 3

Lemma

Let a diagram K_1 be obtained from a diagram K_2 by applying a second Reidemeister move. Then two 2-colored diagrams of the knot obtained by applying the algorithm of obtaining 2-colored diagram to K_1 and K_2 with the same orientations are regular equivalent.

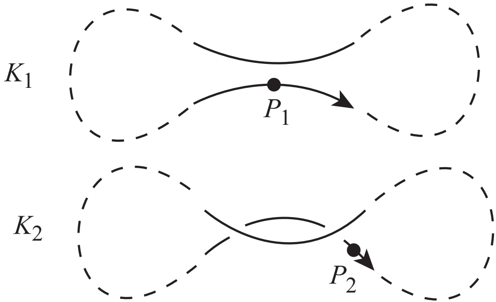


Figure: Two knots K_1 and K_2 with orientations and fixed points P_1 and P_2

Lemma 3

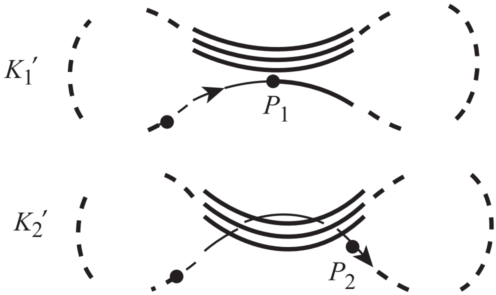


Figure: Two 2-colored diagrams K_1' and K_2'

Lemma 4

Lemma

Let a diagram K_1 be obtained from a diagram K_2 by applying a third Reidemeister move. Then two 2-colored diagrams of the knot obtained by applying the algorithm of obtaining 2-colored diagram to K_1 and K_2 with the same orientations are regular equivalent.

Lemma 4

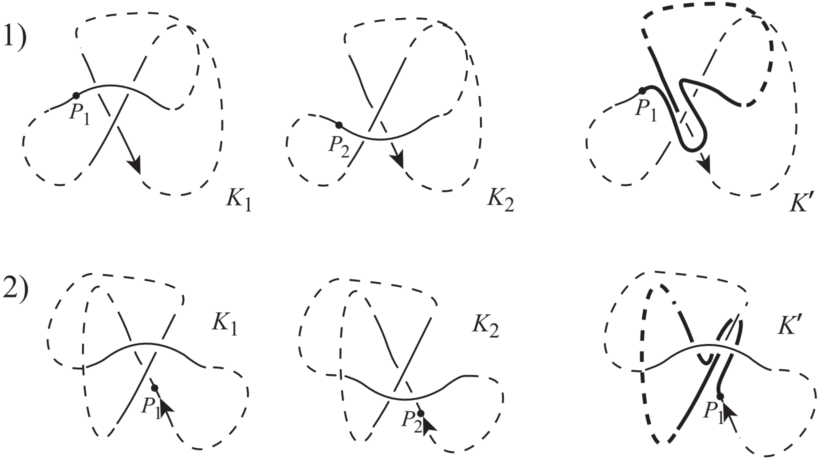


Figure: Constructing 2-colored diagrams

Lemma 4

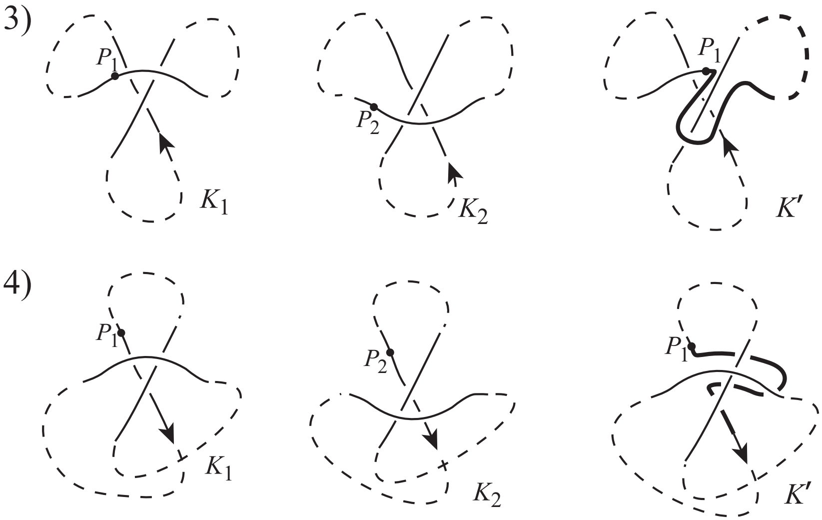


Figure: Constructing 2-colored diagrams

Proof

$\Rightarrow$ Obviously, diagrams are equivalent if they are regularly equivalent.

$\Leftarrow$

$$\begin{array}{ccccccccccccc} K & = & K_1 & \to & K_2 & \to & \dots & \to & K_n & = & \widetilde{K} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ K & \to & K_1' & \to & K_2' & \to & \dots & \to & K_n' & \to & \widetilde{K} \end{array}$$

Realizability of a word

Definition

A closed curve on an oriented surface is called *normal* if it has only finitely many self-intersections and if the self-intersections are transverse double points (i.e., crossings).

Definition

The *unsigned Gauss word* is a word, in some given finite alphabet, in which each letter occurs exactly twice.
A *signed Gauss word* is an unsigned Gauss word in which each letter has a superscript of 1 or -1 and the two occurrences of each letter have superscripts of opposite sign.

Definition

A signed Gauss word is said to be *planar* if it is the intersection sequence of a normal closed curve.

Signed Gauss word

To each normal curve γ we can assign a signed Gauss word. We label all crossings of γ with the letters $a_1, \ldots, a_n$, take any arbitrary point P (not coinciding with crossings) on γ and orient our curve. Traversing γ from P according to the orientation we record symbol a_i^{+1} or a_i^{-1} for each crossing a_i as illustrated on the picture. The result is a word $w = a_{i_1}^{j_1} a_{i_2}^{j_2} \ldots a_{i_{2n}}^{j_{2n}}$, where n is the number of crossings of γ and $a_{i_k} \in \{a_1, \ldots, a_n\}$ and $j_k \in \{\pm 1\}$, $k = 1, \ldots, 2n$.

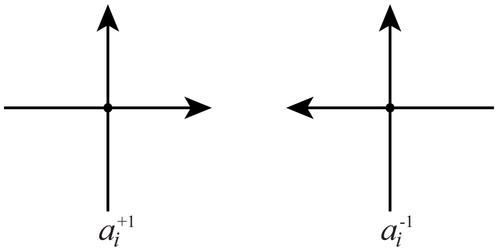


Figure: Superscripts

Example

$$1^{+1}2^{+1}3^{+1}4^{-1}5^{-1}6^{-1}4^{+1}1^{-1}2^{-1}5^{+1}6^{+1}3^{-1}$$

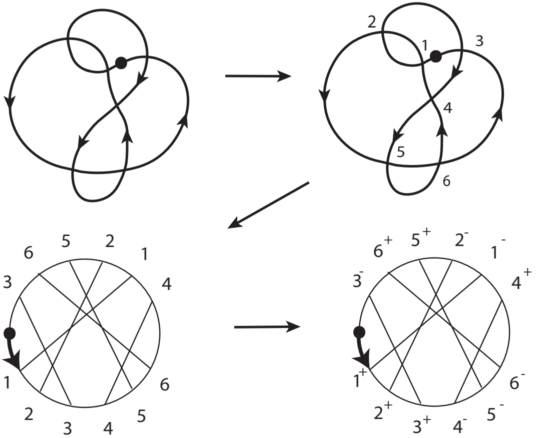


Figure: From a normal curve to a signed Gauss word

Planarity of a signed Gauss word

Let S_i denote the subset of letters that occur in w, in cyclic order, between the letters a_i^{+1} and a_i^{-1}, not including a_i^{+1} and a_i^{-1}; and $\bar{S}_i = S_i \cup \{a_i^{+1}, a_i^{-1}\}$, S_i^{-1} is the set obtained from S_i by changing the signs of the superscripts of the letters of S_i. Define $\alpha_i(w)$ as a sum of the superscripts of the letters of S_i and $\beta_{ij}(w)$ as a sum of superscripts of the letters in the intersection $\bar{S}_i \cap S_j^{-1}$.

Theorem (G. Cairns, D. Elton, 1996)

A signed Gauss word $w = a_{i_1}^{j_1} a_{i_2}^{j_2} \dots a_{i_{2n}}^{j_{2n}}$ is planar if and only if $\alpha_i(w) \equiv 0 \pmod 2$ and $\beta_{ij}(w) \equiv 0 \pmod 2$ for each $i, j \in \{1, \dots, n\}$.

Back to knot diagrams

Corollary

A chord diagram with arrows and signs corresponds to a knot diagram if and only if the corresponding signed Gauss word is planar.

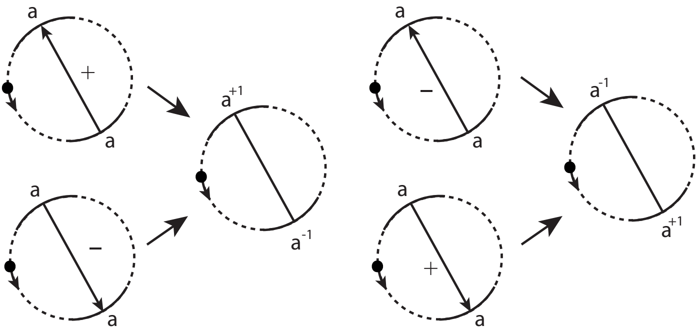


Figure: A signed Gauss word for a knot diagram

Signed Gauss word and Gauss diagram

$$1^{+1}2^{+1}3^{+1}4^{-1}5^{-1}6^{-1}4^{+1}1^{-1}2^{-1}5^{+1}6^{+1}3^{-1}$$

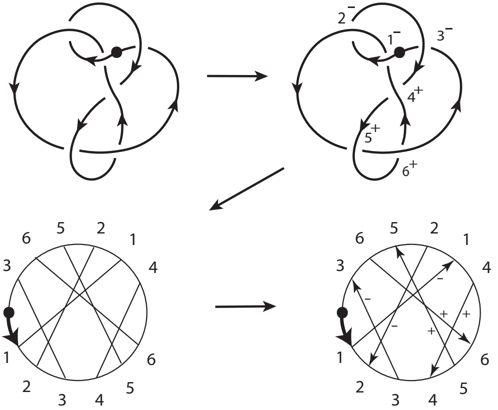


Figure: Gauss diagram

For 2-colored diagrams

Let K be a 2-colored diagram with n crossings. We have two points P_1 and P_2 on K which divide K into two curves. Taking any point of P_i and orienting K we get the Gauss diagram of K. Labeling all crossings with $\{1, \ldots, n\}$ according to the orientation we will get that the first half of the word corresponding to the Gauss diagram will be $1 \ldots n$. Thus, to denote the Gauss diagram we can use only the second half of the word.

2-signed Gauss words

Definition

2-*signed Gauss word* are such words, in some given finite alphabet, in which each letter occurs exactly once and has two types of additional information. Each letter will have a superscript of $+1$ or -1, where $+1$ (resp., -1) stands for letters with the writhe number $+1$ (resp., -1) and an index of u or o, where o (resp., u) stands for letters corresponding to the beginning (resp, the end) of the arrow.

For our example:

$$4_o^{+1}1_u^{-1}2_o^{-1}5_u^{+1}6_o^{+1}3_u^{-1}.$$

Reformulation

Now we want to reformulate theorem about realizability of a signed Gauss word for 2-signed Gauss words by using Corollary. Let $w = k_{1_{r_1}}^{\delta_1} \dots k_{n_{r_n}}^{\delta_n}$ be a 2-signed Gauss word of the length n, $\alpha_j(w) = n - k_j + j - 1$, where $j \in \{1, \dots, n\}$, and β_{pq}, $p < q$, are defined as follows.
If (δ_p, r_p), $(\delta_q, r_q) \in \{(+1, o), (-1, u)\}$ or (δ_p, r_p), $(\delta_q, r_q) \in \{(-1, o), (+1, u)\}$, then

$$\beta_{pq}(w) = \#\{k_l \,|\, k_l \geq k_p,\ l = 1, \dots, q-1\} + \#\{k_l \,|\, k_l > k_q,\ l = 1, \dots, p\},$$

and if $(\delta_p, r_p) \in \{(+1, o), (-1, u)\}$, $(\delta_q, r_q) \in \{(-1, o), (+1, u)\}$ or $(\delta_p, r_p) \in \{(-1, o), (+1, u)\}$, $(\delta_q, r_q) \in \{(+1, o), (-1, u)\}$, then

$$\beta_{pq}(w) = \#\{k_l \,|\, k_l \geq k_p,\ l = q+1, \dots, n\} + \#\{k_l \,|\, k_l < k_q,\ l = 1, \dots, p\},$$

where $\#M$ denotes the number of elements of the set M.

Theorem

A 2-signed Gauss word w corresponds to a 2-colored diagram if and only if $\alpha_i(w) \equiv 0 \pmod 2$ for all i and $\beta_{pq}(w) \equiv 0 \pmod 2$ for all $p < q$.

Thank you for your attention!