Nambu bracket and sigma-models

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July 5th, 2017

Outline

Chern-Simons model

The Poisson sigma-model and its twisted version

The Nambu mechanics

The Nambu σ-models

3d TQFT and knots

Put M -3d manifold with a principial bundle with group G (A -conection form)

$$S_{CS} = \frac{k}{4\pi} \int_M tr(AdA + \frac{2}{3} A \wedge A \wedge A)$$

The partition function

$$Z(k) = \int DAe^{iS_{CS}}$$

Feynman's integral over all equivalence classes of connections modulo gauge transformations
k is a coupling constant and for gauge invariance $k \in \mathbb{Z}$

Here gauge invariant observables are the Wilson lines

$$W_R(C) = tr_R P e^{\oint_C A},$$

R is irreducible representation of G, C is oriented closed curve (knot), P is P-ordering
Put a link C with components K_1,..., K_L

$$W_{R_1,\dots,R_L}(C) = < W_{R_1}^{K_L} \dots W_{R_L}^{K_L} > = \frac{1}{Z(M)} \int DA \left(\prod_{i=1}^{L} W_{R_i}^{K_i} \right) e^{iS_{CS}}$$

SU(N)

For $SU(N)$ one can obtain the HOMFLY-PT polynomial $P_C(q,\lambda)$ (for $SU(2)$ the Jones polynomial) and all $R_i = \Box$

$$W_{\Box,\dots,\Box}(C) \sim P_C(q,\lambda),$$

where $q = \exp(\frac{2\pi i}{k+N})$ and $\lambda = q^N$

$SU(2) \rightarrow$ the Jones polynomials

$SO(N) \rightarrow$ the Kauffman polynomial

The partition functions for SU(N)

The partition function for $SU(N)$

$$Z(S^3, N, k) = \frac{1}{\sqrt{N}(k+N)^{\frac{N}{2}-1}} \prod_{i=1}^{N-1} (2\sin(\frac{\pi i}{k+N}))^{N-i}$$

$Z(T^3, N, k)$ counts the number of integrable irreducible highest-weight representations of the affine Kac-Moody algebra $SU(N)_k$

$$Z(T^3, N, k) = \frac{1}{B(N, K)}$$

The Courant sigma-model (D. Roytenberg '01)

Put some vector bundle $E \to M_0$. Let us consider supermanifold ΠT^*E. If $\{x_i\}$ are local coordinates on M_0 and $\{e_i\}$ is a local basis of section for E with $< e_a, e_b >= g_{ab} = const$.
Courant algebroid $(E, < \cdot, \cdot >)$ the anchor map has matrix P^i_a

$$S(X, A, F) = \int_N F_i dX^i + \frac{1}{2} A^a g_{ab} dA^b - A^a P^i_a(X) F_i + \frac{1}{6} T_{abc} A^a A^b A^c + \ldots$$

$F = 0$ and $T_{abc} = \epsilon_{abc}$ one can obtain usual Chern-Simons theory

The Poisson manifold

Put N is a Poisson manifold
(N has Poisson bi-vector $\pi = \frac{1}{2}\pi^{ij}(x)\partial_i \wedge \partial_j$)

$$\{f(x), g(x)\} = \pi^{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

The Jacobi identity $\{\{f, g\}, h\} + cycl. = 0$

$$[\pi, \pi]_{NS} = 0$$

The time evolution is given by

$$\frac{df}{dt} = \{H, f\}$$

The Schouten-Nijenhuis bracket

The bracket for polivectors

$$[\cdot,\cdot]_{NS}: M^n(N)\times M^m(N)\to M^{n+m-1}(N),$$

$$[u_1\wedge\ldots\wedge u_n, v_1\wedge\ldots\wedge v_m]_{NS} = \sum_{i,j}(-1)^{i+j}[u_i, v_j]\wedge\ldots\wedge \not{u_i}\wedge\ldots v_1\wedge\ldots \not{v_j}\wedge\ldots v_n$$

2d σ-models

Usually 2d sigma-models have action

$$S = \int_{\Sigma} \eta^{\mu\nu}(x) g^{ij}(\phi) \partial_\mu \phi_i \partial_\nu \phi_j \sqrt{\eta(x)} d^2 x$$

here Σ is two-dimensional (super)manifold (world-sheet or space-time) with metric η and fields ϕ_i lie in a (super)manifold L (target space) with metric g
Simple example $O(N)$-model

$$S = \frac{1}{2g^2} \int_X \partial_\mu \phi_i \partial_\mu \phi_i, \quad \phi_i \phi_i = 1$$

Supersymmetric examples are A and B-models

The Poisson σ-model

Put (L, π) the Poisson manifold with fields 0-forms X^i (cordinates on target L) and 1-form A

$$A = A_{\mu i} dx^{\mu} \wedge dX^i$$

Then the topological Poisson σ-model

$$S_t = \int_{\Sigma} A_i \wedge dX^i + \frac{1}{2}\pi^{ij}(X) A_i \wedge dA_j$$

Seiberg-Witten map

$$S_1 = \int_\Sigma A_i \wedge dX^i + \frac{1}{2}\pi^{ij} A_i \wedge A_j + \frac{1}{2}(G^{-1})^{ij} A_i \wedge *A_j$$

Put $2\pi\alpha' = 1$. The string sigma-model is

$$S_2 = \frac{1}{2}\int_\Sigma (g_{ij} dX^i \wedge *dX^j + B_{ij} dX^i \wedge dX^j)$$

g and B are related to G^{-1} and π

$$\frac{1}{g+B} = G^{-1} + \pi$$

Topological limit $G^{-1} \to 0$
Noncommutative limit $g \to 0 (\alpha' \to 0)$

$$x_i \to \tilde{x}_i = x_i + \theta_{ij} A_j$$

twisted Poisson structure

If (N,π) has a closed 3-form H. Then the twisted Jacobi identity

$$\frac{1}{2}[\pi,\pi]_{SN} = \Lambda^3 \pi H$$

$$S(X,A) = \int_\Sigma (A_i \wedge dX^i + \frac{1}{2}\pi^{ij} A_i \wedge A_j) + S_{WZ},$$

$$S_{WZ} = \int_\Lambda \tilde{H},$$

where $\partial\Lambda = \Sigma$ and $\tilde{H}$ mean the pullback of H to Λ by extinsion to V of the map X^i

n-Lie algebras

an algebra with n-ary map $[] : A^{\wedge n} \to A$ (Nambu-Filippov algebra)

$$[L_1, L_2, ..., L_n] = (-1)^{\epsilon(\sigma)}[L_{\sigma_1}, ...L_{\sigma_n}] \quad \sigma \in S_n$$

$$[L_{i_1}, L_{i_2}, ..., L_{i_n}] = f^k_{i_1...i_n} L_k$$

The Leibniz rule

$$[L_0 L_{i_1}, L_{i_2}, ..., L_{i_n}] = L_0[L_{i_1}, L_{i_2}, ..., L_{i_n}] + [L_0, L_{i_2}, ..., L_{i_n}]L_{i_1}$$

There is generalized Jacobi identity (Fundamential identity)

Example. Nambu-Heisenberg algebra

$$[x_1, x_2, x_3] = x_1 x_2 x_3 - x_1 x_3 x_2 + x_3 x_1 x_2 - x_3 x_2 x_1 + x_2 x_3 x_1 - x_2 x_1 x_3 = -i\hbar I$$

The Nambu bracket

$$\{f_1, f_2, ..., f_n\} = \pi^n(df_1, df_2, ...df_n),$$

$$\pi^n = \pi^{i_1 \dots i_n} \partial_{i_1} \wedge ... \wedge \partial_{i_n}$$

Generalized Jacobi identity (Fundamential identity)

$$\{\{f_1, f_2, ..., f_n\}, f_{n+1}, ..., f_{2n-1}\} + \{f_n, \{f_1, f_2, ..., f_n\}, f_{n+2}, ..., f_{2n-1}\} + ...$$

$$+\{f_{n+1}, ...f_{2n-1}, \{f_1, f_2, ..., f_n\}\} = \{f_1, f_2, ..., f_{n-1}, \{f_n, ..., f_{2n-1}\}\}$$

$$\frac{df}{dt} = \{H_1, H_2, ..., H_n, f\}$$

Example of Nambu mechanics

The asymmetric Euler top with $H_1 = \frac{1}{2}(\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3})$ and
$H_2 = \frac{1}{2}(L_1^2 + L_2^2 + L_3^2)$
The Nambu equations $\frac{dL_i}{dt} = \epsilon^{ijk}\partial_j H_1 \partial_k H_2$

$$\frac{dL_1}{dt} = \left(\frac{1}{I_2} - \frac{1}{I_3}\right) L_2 L_3, \quad \frac{dL_2}{dt} = \left(\frac{1}{I_3} - \frac{1}{I_1}\right) L_3 L_1, \frac{dL_3}{dt} = \left(\frac{1}{I_2} - \frac{1}{I_3}\right) L_1 L$$

3d Nambu topological sigma-model (Schupp, Jurco '2012)

Put 1-forms $A_{ij} = A_{ij\alpha} dx^\alpha$ and 2-forms $B_i = B_{i\alpha\beta} dx^\alpha \wedge dx^\beta$ on N and $\alpha, \beta = 0, 1, 2$
The action for the topological Nambu sigma-model has form

$$S_N(X, A, B) = \int_N (B_i \wedge dX^i + A_{ij} \wedge dX^i dX^j + \frac{1}{6}\pi^{ijk} A_{ij} \wedge B_k)$$

Here one can add terms which depends on metric

$$\frac{1}{2}\int_N (G^{ij} B_i \wedge *B_j + \tilde{G}^{klmn} A_{kl} \wedge *A_{mn})$$

There is a similar description by the membrane sigma model and Seiberg-Witten map

Remark about twisted Nambu bracket and sigma-model

It ia possible to imagine the Jacobi identity for the Nambu bracket like usual the Poisson bracket

$$[\pi^n, \pi^n]_{SN} = 0$$

If manifold V ($N = \partial V$) has some $(n+1)$-form H we again can add right side in the Jacobi identity and Wess-Zumino term in the action

$$[\pi^n, \pi^n]_{SN} = \Lambda^3 \pi^n H.$$

$$S = S_N + \int_V \tilde{H}$$

Thank you for attention