Representations of virtual braid groups to rook algebras and virtual links invariants

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Braid group on n strands, denoted by $\mathcal{B}_n$, is a group generated by $\sigma_1, \ldots, \sigma_{n-1}$ satisfying the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i,$$  
if $|i - j| > 1$,

$$\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i,$$  
for $i = 1, \ldots, n - 2$. 

Fix the points $P_i = (i, 1)$ and $Q_i = (i, 0)$ in $\mathbb{R}^2$ for $i = 1, 2, \ldots, n$. For braid word $\omega$, presented braid $\beta \in \mathcal{B}_n$, we connect $P_i$ and $Q_j$ by drawing following diagrams

![Diagram](image)

for generators $\sigma_i$ and $\sigma_i^{-1}$. The result is called the diagram of braid $\beta$.
Relations of braid groups correspond to plane isotopies and Reidemeister moves 2 and 3.

The geometrical interpretation of relation $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$.

The set of all braid diagrams up to isotopies and Reidemeister moves form a group, isomorphic to braid group $B_n$. 
Closure of the braid $\beta$ is the link, that can be obtained of geometric representative of braid $\beta$ by identifying $Q_i$ and $P_i$ for $i = 1, \ldots, n$. 
Theorem (J.Alexander)
Every link can be represented as a closed braid.

Theorem (A.Markov)
Two braids $\beta_1 \in B_n, \beta_2 \in B_m$ has the same closures if and only if $\beta_2$ can be obtained from $\beta_1$ by sequence of following moves or its inverses:

1. $\alpha \rightarrow \sigma_i^{-1}\alpha\sigma_i$,
2. $\alpha \rightarrow \ell(\alpha)\sigma_n^{\pm1}$,

here $\alpha, \sigma_i \in B_n, \sigma_n \in B_{n+1}$ and $\ell$ is a natural embedding of $B_n$ to $B_{n+1}$.

Now we can consider links as braids up to Markov moves.
Virtual braid group on $n$ strands, denoted by $V\mathcal{B}_n$, is a group with generators:

$$\sigma_1, \ldots, \sigma_{n-1}, \rho_1, \ldots, \rho_{n-1}$$

and relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ if } |i - j| > 1,$$
$$\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i, \text{ for } i = 1, \ldots, n - 2,$$

$$\rho_i^2 = e, \text{ for } i = 1, \ldots, n - 1,$$
$$\rho_{i+1} \rho_i \rho_{i+1} = \rho_i \rho_{i+1} \rho_i, \text{ for } i = 1, \ldots, n - 2,$$
$$\rho_i \rho_j = \rho_j \rho_i, \text{ if } |i - j| > 1,$$

$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, \text{ for } i = 1, \ldots, n - 2,$$
$$\sigma_i \rho_j = \rho_j \sigma_i, \text{ if } |i - j| > 1.$$
The diagrammatic interpretation of generator $\rho_i$.

Closure of virtual braid is defined similarly as closure of classical braid.
Let $R_n, n \geq 1$ denote a set of $n \times n$ matrices with entries from the set $\{0, 1\}$ having at most one 1 in each row and in each column.

Example for $n = 2$

\[
\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}
\]

$R_n$ with the standard matrix multiplication is monoid, called a rook monoid.
Rook diagram is a bipartite graph with $n$ vertices in each partite, such that each vertex has degree either zero or one. We will draw one partite on the top and another on bottom of a rectangle.

There is one-to-one correspondence between rook diagrams and matrices of $R_n$. 
Let $d_1$ and $d_2$ be rook diagrams with the same number $2n$ of vertices. The product $d_1 d_2$ is a rook diagram with $2n$ vertices and edges, defined by the rule presented at the following picture.

Set of all diagram with this geometrical defined multiplication is monoid, isomorphic to $R_n$.
Given diagrams $d_1$ and $d_2$, we define the tensor product, denoted $d_1 \otimes d_2$, to be the result of appending of $d_2$ to the right of $d_1$. 
Diagram from $R_n$ is said to be planar if it can be drawn (keeping inside of the rectangle formed by its vertices) without any crossings of edges.

Denote by $P_n$ the set of all planar diagrams of $R_n$. It is easy to see that $P_n$ is a submonoid of $R_n$. 
A rook algebra, denoted by $CR_n$, is a $C$-algebra generated by $R_n$.

A planar rook algebra, denoted by $CP_n$, is a $C$-algebra generated by $P_n$. 
We denote elements of $\mathbb{R}_2$ as following:

\[
\begin{align*}
    d_1 &= \begin{array}{c}
    \bullet \\
    \bullet \\
    \bullet \\
    \bullet
    \end{array} \\
    d_2 &= \begin{array}{c}
    \bullet \\
    \bullet
    \end{array} \\
    d_3 &= \begin{array}{c}
    \bullet \\
    \bullet \\
    \bullet
    \end{array} \\
    d_4 &= \begin{array}{c}
    \bullet \\
    \bullet \\
    \bullet
    \end{array} \\
    d_5 &= \begin{array}{c}
    \bullet \\
    \bullet
    \end{array} \\
    d_6 &= \begin{array}{c}
    \bullet \\
    \bullet
    \end{array} \\
    d_7 &= \begin{array}{c}
    \bullet \\
    \bullet
    \end{array}
\end{align*}
\]
Define mapping $\varphi : \mathbb{B}_n \to \mathbb{C}P_n$ by the following rule:

$$\varphi(\sigma_i) = a \cdot d_{1i} + b \cdot d_{2i} + c \cdot d_{3i} + d \cdot d_{4i} + e \cdot d_{5i} + d_{6i}$$

where

$$d_{ji} = I^\otimes i - 1 \otimes d_j \otimes I^\otimes n - i - 1, \quad a, b, c, d, e \in \mathbb{C} \text{ and } I \text{ is the identity diagram in } P_1.$$
Theorem (S. Bigelow, E. Ramos, R. Yi)

Assuming \( a + c + d \neq 1 \) and \( cd \neq 0 \), a mapping of the above form is a homomorphism if and only if its coefficients are in one of the following families:

1. \( b = e = -1 \),
2. \( a = -c - d, b = -1, e = -cd \),
3. \( a = -c - d, b = -cd, e = -1 \),
4. \( a = 1 - c - d + cd, b = -cd, e = -1 \),
5. \( a = 1 - c - d + cd, b = -1, e = -cd \).
Define mapping $\psi_k : \mathcal{VB}_n \rightarrow \mathbb{CR}_n$ by the following rule:

$$\psi_k(\sigma_i) = \varphi_k(\sigma_i)$$
$$\psi_k(\rho_i) = d_{i,7}$$

$$d_7 = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}$$

**Theorem 1**
The mapping $\psi_k$ is a representation of $\mathcal{VB}_n$ for any $k = 1, \ldots, 5$. 
Example 1

Let $\psi^2,^3_5$ be the particular case of $\psi_5$ for $c = 2, d = 3$. It is known that the braid $\beta = (\sigma_1^2 \rho_1 \sigma_1^{-1} \rho_1 \sigma_1^{-1} \rho_1)^2 \in VB_2$ cannot be distinguished from the trivial by the Burau presentation. Direct computations show that

$$\psi^2,^3_5 \left((\sigma_1^2 \rho_1 \sigma_1^{-1} \rho_1 \sigma_1^{-1} \rho_1)^2\right) =$$

$$- \frac{2200}{9} d_1 - \frac{500}{27} d_2 - \frac{2450}{27} d_3 + \frac{1550}{27} d_4 + \frac{8000}{27} d_5 + d_6,$$

so $\psi^2,^3_5$ distinguish it from the trivial braid.
\[ \dim(\mathbb{CP}_n) = |P_n| = \sum_{k=0}^{n} \binom{n}{k}^2 \]

For \( n = 1, 2, 3, 4, 5, 6 \) we get 2, 6, 20, 70, 252, 924.

\[ \dim(\mathbb{CR}_n) = |R_n| = \sum_{k=0}^{n} \binom{n}{k}^2 k! \]

For \( n = 1, 2, 3, 4, 5, 6 \) we get 2, 7, 34, 209, 1546, 13327.
Let $[\ ] : \mathbb{CR}_n \rightarrow M_n(\mathbb{C})$ be a linear mapping, defined for any $d \in R_n$ as matrix, corresponding to diagram $d$.

Considering coefficient $c$ as variable, we define mapping $\phi : V\mathcal{B}_n \rightarrow GL_n(\mathbb{Z}[c^{\pm 1}])$ by the following rule:

$$
\phi(\sigma_i) = -\frac{1}{c^2} [\psi_2(\sigma_i)] \bigg|_{d=-c} = \begin{pmatrix}
I_{i-1} & 0 & \frac{1}{c} & \frac{1+c^2}{c^2} \\
0 & -\frac{1}{c} & \frac{1}{c^2} & I_{n-i-1}
\end{pmatrix}
$$

$$
\phi(\rho_i) = [\psi_2(\rho_i)] = \begin{pmatrix}
I_{i-1} & 0 & 1 \\
1 & 0 & I_{n-i-1}
\end{pmatrix}
$$
Theorem (L.Kauffman, S.Lambropoulou)

Two oriented virtual links are isotopic if and only if any two corresponding virtual braids differ by a finite sequence of braid relations $\mathcal{VB}_\infty$ and the following moves or their inverses:

1. $\rho_i \alpha \rho_i \leftarrow \alpha \rightarrow \sigma_i^{-1} \alpha \sigma_i$,
2. $\ell(\alpha) \rho_n \leftarrow \alpha \rightarrow \ell(\alpha) \sigma_n^{\pm 1}$,
3. $\alpha \rightarrow \ell(\alpha) \sigma_n^{-1} \rho_{n-1} \sigma_n$,
4. $\alpha \rightarrow \ell(\alpha) \rho_n \rho_{n-1} \sigma_{n-1} \rho_n \sigma_n^{-1} \rho_{n-1} \rho_n$,

where $\alpha, \rho_i, \sigma_i \in \mathcal{VB}_n$, $\rho_n, \sigma_n \in \mathcal{VB}_{n+1}$ and $\ell$ is a natural embedding of $\mathcal{VB}_n$ to $\mathcal{VB}_{n+1}$. 
For a virtual braid $\alpha \in V\mathcal{B}_n$ denote $F(\alpha)$ polynomial det$(I_n - \phi(\alpha)) \in \mathbb{Z}[c^{\pm 1}]$.

**Theorem 2**
Let $\alpha \in V\mathcal{B}_n$. For the Kauffman-Lambropoulou move

$$\alpha \rightarrow \ell(\alpha)\sigma_n^{-1}$$

we have

$$F(\alpha) = \left(-\frac{1}{c^2}\right) F(\ell(\alpha)\sigma_n^{-1}).$$

For all other Kauffman-Lambropoulou moves $F(\alpha)$ keeps invariant.

**Corollary**
Let $\alpha_1 \in V\mathcal{B}_n$ and $\alpha_2 \in V\mathcal{B}_m$ correspond to the same virtual link, then $F(\alpha_1) = \left(-\frac{1}{c^2}\right)^k F(\alpha_2)$ for some $k \in \mathbb{Z}$. 
Example of calculation $F(\beta)$

Consider $\beta = \sigma_1 \rho_1 \in V \mathcal{B}_2$.

$$\phi(\sigma_1 \rho_1) = \begin{pmatrix} 0 & \frac{1}{c} \\ \frac{1}{c} & \frac{1+c^2}{c^2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ \frac{1+c^2}{c^2} & \frac{1}{c} \end{pmatrix},$$

$$F(\sigma_1 \rho_1) = \det \begin{pmatrix} -\frac{1}{c} + 1 & 0 \\ -\frac{1+c^2}{c^2} & -\frac{1}{c} + 1 \end{pmatrix} = \frac{(c - 1)^2}{c^2} = \frac{1}{c^2} - \frac{2}{c} + 1.$$
Theorem (T. Kadokami)

Let \( \alpha = \prod_{i=1}^{m} \sigma_{1}^{q_{i}} \rho_{1} \), \( \beta = \prod_{i=1}^{k} \sigma_{1}^{p_{i}} \rho_{1} \), for some \( k, l, q_{i}, p_{i} \in \mathbb{Z} \) such that \( l, k \geq 1 \) and \( q_{i}, p_{i} \neq 0 \). If \( \alpha \) and \( \beta \) correspond to the same virtual link then \( \alpha \) and \( \beta \) are conjugated in \( VB_{2} \).

Let \( \kappa(d) \) be a number of vertical lines in diagram \( d \in R_{n} \), \( f \) – some function, defined on integers. Define linear map \( \text{tr}_{f} : R_{n} \to \mathbb{R} \) by following equality

\[
\text{tr}_{f}(d) = f(\kappa(d)).
\]

Notice

Function \( \kappa : CR_{n} \to \mathbb{N} \) is commutative, so \( \text{tr}_{f} \) is commutative too.
Let $t \in \mathbb{C}$ be a complex variable, define linear mapping $\partial : \mathbb{CR}_2 \to \mathbb{CR}_2$ assuming that:

$$
\begin{align*}
\partial(d_1) &= \partial(d_6) = \partial(d_7) = 0, \\
\partial(d_2) &= t(d_3 - d_4) = -\partial(d_5), \\
\partial(d_3) &= t(d_2 - d_5) = -\partial(d_4).
\end{align*}
$$

**Theorem 3**

Mapping $\partial : \mathbb{CR}_2 \to \mathbb{CR}_2$ is a derivation on $\mathbb{CR}_2$, i.e. it satisfies the Leibniz relation

$$
\partial(D_1D_2) = \partial(D_1)D_2 + D_1 \partial(D_2).
$$

for any $D_1, D_2 \in \mathbb{CR}_2$. 
Lemma
Let \( F \) – commutative linear function on \( \mathbb{C}R_2 \), then composition \( F \circ \partial \) is commutative.

For virtual braid \( \beta \in \mathcal{VB}_2 \) and integer \( m \in \mathbb{Z} \) associate the value
\[
T_f \circ \partial^m(\beta) = \text{tr}_f(\partial^m(\psi(\beta))).
\]

Theorem 4
Let \( \alpha, \beta \in \mathcal{VB}_2 \) be braids satisfying conditions of Kadokami theorem, then for any integer \( m \geq 0 \) and any function \( f \) we have
\[
T_f \circ \partial^m(\beta) = T_f \circ \partial^m(\alpha).
\]
Example 2

Consider values $T_f$ and $T_f \circ \partial$ with $f(\pi) = \pi$. It is easy to see, that $\beta_1 = \sigma_1^3 \rho_1 \sigma_1^2 \rho_1 \sigma_1 \rho_1$ and $\beta_2 = \sigma_1^3 \rho_1 \sigma_1 \rho_1 \sigma_1^2 \rho_1$ are not conjugated in $V\mathcal{B}_2$. We have

$$T_f(\beta_1) = T_f(\beta_2),$$

but

$$T_f \circ \partial(\beta_1) \neq T_f \circ \partial(\beta_2).$$

Thus, the derivation $\partial$ allows us to distinguish more virtual links.
Thank you for attention!