The chord index and its applications

Zhiyun Cheng

Beijing Normal University

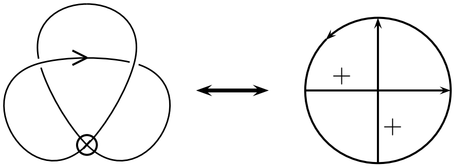
4th Russian-Chinese Conference on Knot Theory and Related Topics

Bauman Moscow State Technical University   July 4, 2017

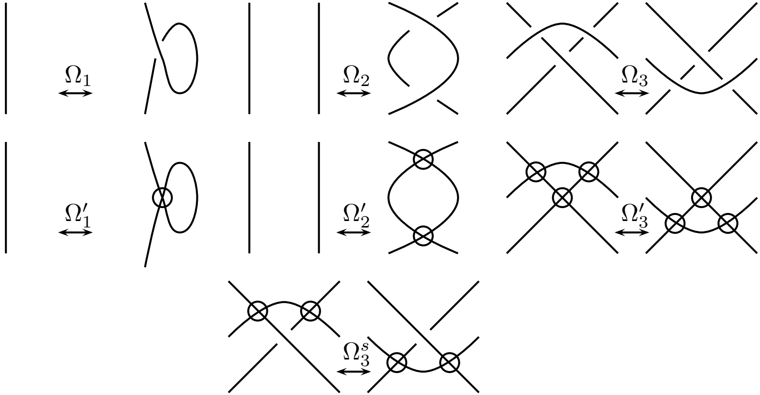
Virtual knot theory and chord index

A virtual knot (L. Kauffman 1999) is

- an embedding of S^1 in $\Sigma_g \times [0,1]$ up to isotopy and stabilizations
- a chord diagram up to Reidemeister moves



Virtual link types= {all virtual link diagrams}/{generalized Reidemeister moves}



A (naive) question is

Question

Can we define a virtual knot invariant by counting the number of chords in a chord diagram (number of crossings in a knot diagram)?

Unfortunately, the answer is No.

Actually, we know that the signed sum (writhe) is not a virtual knot invariant.

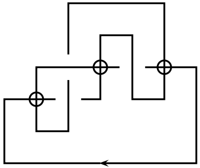
We may ask

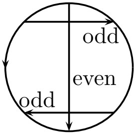
Question

How can we define a virtual knot invariant by counting the number of chords in a chord diagram (number of crossings in a knot diagram)?

A $\mathbb{Z}_2$-assignment

A chord (and the corresponding crossing) is called even/odd if there are an even/odd number of chords which have nonempty intersection with it.





Theorem (L. Kauffman 2004)

The odd writhe $J(K) = \sum_{c_i \in Odd(K)} w(c_i)$ *is a virtual knot invariant, here* $Odd(K)$ *denotes the set of odd crossings and* $w(c_i)$ *is the writhe of* c_i*.*

Remark If K is a classical knot diagram, then $Odd(K) = \emptyset$.

A $\mathbb{Z}$-assignment via Gauss diagram

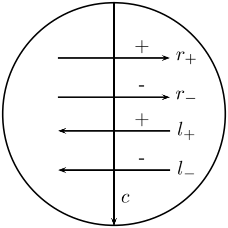
Let $G(K)$ be a Gauss diagram, and c a chord, we define

$r_+(c)$= the number of positive chords crossing c from left to right;

$r_-(c)$= the number of negative chords crossing c from left to right;

$l_+(c)$= the number of positive chords crossing c from right to left;

$l_-(c)$= the number of negative chords crossing c from right to left.



Define the index of c as

$$\mathrm{Ind}(c) = r_+ - r_- - l_+ + l_-.$$

Remarks

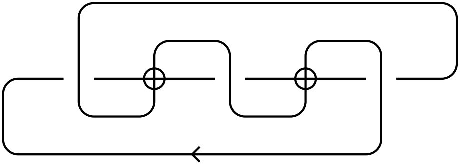
1. The first kind of this index, the "intersection index", was introduced by A. Henrich in 2010, which equals $|\mathrm{Ind}(c)|$.

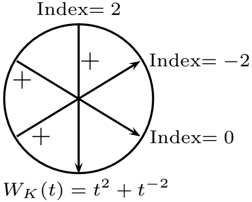
2. In 2013, H. Dye defined a "parity mapping" from the set of chords to $\mathbb{Z}$, which is exactly the inverse of $\mathrm{Ind}(c)$.

Several (mutually equivalent) polynomial invariants of virtual knots have been defined independently:

1. (Gao, C. 2013) The writhe polynomial $W_K(t) = \sum_{\mathrm{Ind}(c_i)\neq 0} w(c_i)t^{\mathrm{Ind}(c_i)}$ is a virtual knot invariant.
2. (H. A. Dye 2013) The Gauss diagram invariant $|A_n(G(K))|$ equals the coefficient of t^{-n} in $W_K(t)$.
3. (Y. H. Im, S. Kim, D. S. Lee 2013) The parity writhe polynomial $F_K(x, y) =$
$$\sum_{\mathrm{Ind}(c_i)\text{ is odd}} w(c_i)x^{\mathrm{Ind}(c_i)+1} + \sum_{\mathrm{Ind}(c_i)\text{ is even}} w(c_i)y^{\mathrm{Ind}(c_i)+1} - w(K)x.$$
By replacing y with x, it coincides with $(W_K(x) - W_K(1))x$.
4. (L. Folwaczny, L. Kauffman 2013) The affine index polynomial can be described as $P_K(t) = W_K(t) - W_K(1)$.
5. (S. Satoh, K. Taniguchi 2014) The nth parity writhe $J_n(K)$ is equal to the coefficient of t^n in $W_K(t)$.

An example:





Proposition

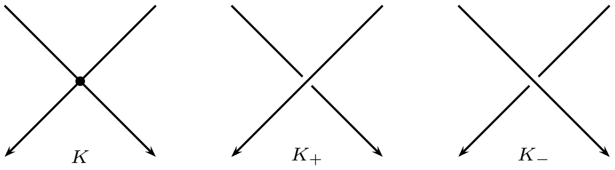
1. *$W_{r(K)}(t) = W_K(t^{-1}), W_{m(K)}(t) = -W_K(t^{-1})$, here $r(K)$ is the inverse of K and $m(K)$ is the mirror image of K.*
2. *Assume $W_K(t) = \sum_n a_n t^n$, then $|a_n|$ provides a lower bound of the number of crossing points with index n for any $n \neq 0$.*
3. *(B. Mellor 2016) $\mathrm{span} W_K(t) \leq 2c_v(K)$, here $c_v(K)$ denotes the virtual crossing number of K.*

Finite type invariant of virtual knots

Finite type invariant (Vassiliev invariant)

- f is a knot invariant which take values in an abelian group.
- Extend f to an invariant of singular knots with n singularities via the following recursive relation.

$$f^{(n)}(K) = f^{(n-1)}(K_+) - f^{(n-1)}(K_-)$$



- f is a finite type invariant of degree n if it vanishes on singular knots with $n+1$ singularities and does not vanish on some singular knot with n singularities.

Finite type virtual knot invariant of degree 0

- Finite type invariant of degree 0 $\Rightarrow$ it is invariant under crossing change $\Rightarrow$ it take the same value on all classical knots.
- (Sawollek 2003, Henrich 2010) For virtual knots, finite type invariant of degree 0 need not to be trivial.

Recall that the writhe polynomial $W_K(t) = \sum_{\mathrm{Ind}(c_i) \neq 0} w(c_i) t^{\mathrm{Ind}(c_i)}$, we define

$$\mathfrak{F}_K(t) = W_K(t) - W_K(t^{-1}).$$

Theorem (C. 2016)

$\mathfrak{F}_K(t)$ *is a finite type virtual knot invariant of degree 0.*

Finite type virtual knot invariant of degree 1

Rewrite the writhe polynomial

$$W_K(t) = \sum_{\mathrm{Ind}(c_i)\neq 0} w(c_i)t^{\mathrm{Ind}(c_i)} = \sum_{n=-\infty}^{+\infty} a_n t^n.$$

Note that only finitely many $a_n \neq 0$.

Theorem (Dye 2013)

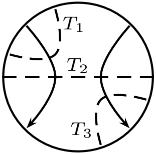
a_n is a finite type virtual knot invariant of degree 1.

Corollary

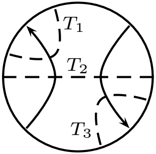
The writhe polynomial $W_K(t)$ is a finite type virtual knot invariant of degree 1.

Finite type virtual knot invariant of degree 2

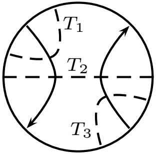
- $G(K)$ is a Gauss diagram.
- $\mathfrak{c}$ denotes a pair of nonintersecting chords.
- Define T to be the set of chords which have nonempty intersection with $\mathfrak{c}$.
- Denote $T = T_1 \cup T_2 \cup T_3$. Each T_i provides an "index" t_i to the pari $\mathfrak{c}$.



(i, k, j)

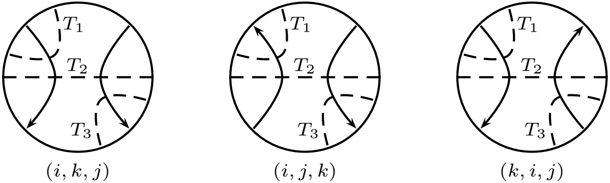


(i, j, k)



(k, i, j)

- Define the triple-index of $\mathfrak{c}$ to be $(|t_1|, |t_2|, |t_3|)$.
- For a fixed triple (i, j, k) ($i, j, k \geq 0$ and $i \neq j \neq k \neq i$), define a set $F_{i,j,k}$ to be all the pairs of nonintersecting chords such that the triple-index of each pair agrees with one the three cases below



Theorem (Chrisman, Dye 2014)

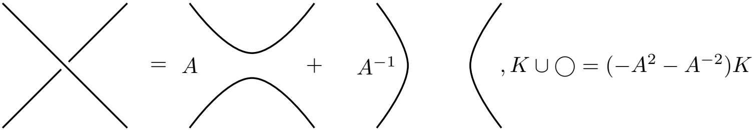
$\phi(K) = \sum_{\mathfrak{c} \in F_{i,j,k}} w(\mathfrak{c}) x^i y^j z^k$ *is a finite type virtual knot invariant of degree 2, where* $w(\mathfrak{c})$ *denotes the product of the writhes of the two chords in* $\mathfrak{c}$.

Indexed Jones polynomial

Given a virtual knot diagram K, for any $n \in \{0, 1, 2, \cdots\}$ we set

$$C_r^n(K) = \{x \in C_r(K) | \mathrm{Ind}(x) \in \{\cdots, -2n, -n, 0, n, 2n, \cdots\}\},$$

here $C_r(K)$ denotes the set of real crossings in K.



After smoothing all crossings in C_r^n, we define $V_K^n(t) =$

$$(-A^{-3})^{w(K)} \sum_s A^{\sharp\text{ 0-smoothing}-\sharp\text{ 1-smoothing}} (-A^2 - A^{-2})^{|s|-1} \Big|_{A=t^{-\frac{1}{4}}}.$$

Theorem (C. 2016)

$V_K^n(t)$ is a virtual knot invariant.

Remark

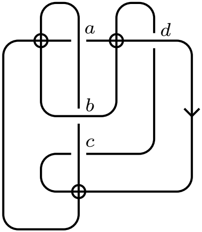
- V_K^1 is nothing but the classical Jones polynomial.
- With some modification (replace $x \notin C_r^n$ with a dot) one can define a more general a polynomial invariant with graphical coefficients. In this case, V_K^2 is essentially equivalent to the parity bracket polynomial defined by V. O. Manturov in 2010.
- Similar idea can be used to define indexed Miyazawa/arrow polynomial.

$c_r^n(K)$ = the minimal number of real crossing points with index n among all diagrams of K.

Proposition (C. 2016)

$c_r^n(K) \geq span V_K^n(t)$.

Example $\mathrm{Ind}(a) = 0$, $\mathrm{Ind}(b) = 1$, $\mathrm{Ind}(c) = 0$, $\mathrm{Ind}(d) = -1$.



- $W_K(t) = t^1 + t^{-1} \Rightarrow c_r^1(K) = c_r^{-1}(K) = 1$.
- $V_K^0(t) = -t^4 + t^3 + t^{\frac{5}{2}} \Rightarrow c_r^0(K) = 2$.

Virtual knot invariant from indexed quandle

An indexed quandle is a set X with a family of binary operations $*_i : X \times X \to X$ $(i \in \mathbb{Z})$ such that

1. $a *_0 a = a$
2. $\forall b, c \in X, i \in \mathbb{Z}$, $\exists! a$ such that $a *_i b = c$
3. $(a *_i b) *_j c = (a *_j c) *_i (b *_{j-i} c)$

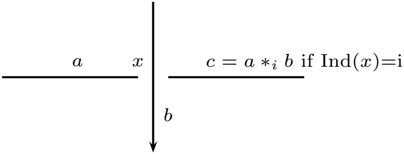
Remark Indexed quandle $\neq$ G-quandle or multi-shelf in general.

Some Examples

- A quandle $(X, *)$ can be naturally thought of as an indexed quandle by defining $*_i = *$ for any $i \in \mathbb{Z}$.
- A quandle $(X, *)$ also can be regarded as an indexed quandle by defining $*_0 = *$ and $a *_i b = a$ for $i \neq 0$.
- Let G be a group, for any $\phi \in Aut(G)$ and $z \in Z(G)$ (the center of G), G can be regarded as an indexed quandle with operations

$$a *_i b = \phi(ab^{-1})bz^i.$$

Let K be a virtual knot diagram, we define the indexed knot quandle $IndQ(K)$ to be the indexed quandle generated by each arc, and each classical crossing gives rise to a certain relation as below



Theorem (C. 2016)

The indexed knot quandle is a virtual knot invariant.

Corollary

Given a finite indexed quandle X, $|Hom(IndQ(K), X)|$ is a virtual knot invariant.

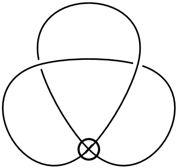
Example

$K =$ virtual trefoil knot.

$X = \{0, 1\}$ with operations $a *_i b = a + i$ (mod 2).

Then

$$|\mathrm{Hom}(IndQ(K), X)| = 2.$$



Note that the virtual trefoil knot has trivial knot quandle.

Cocycle invariant

Let A be an abelian group, if $\phi_i : X \times X \to A$ ($i \in \mathbb{Z}$) satisfy

$$\phi_{j+k}(a *_k b, c) + \phi_k(a, b) = \phi_{j+k}(a, c) + \phi_k(a *_{j+k} c, b *_j c),$$

and in addition

$$\phi_0(a, a) = 1,$$

for any $a, b, c \in X$ and $j, k \in \mathbb{Z}$. Then one can similarly define an invariant as follows

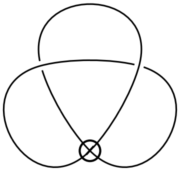
$$\Phi_\phi(K) = \sum_\rho \prod_x \phi_{\mathrm{Ind}(x)}(a, b)^{w(x)},$$

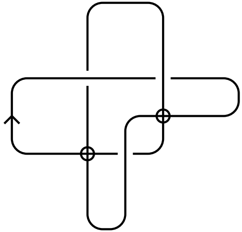
where $\rho \in \mathrm{Hom}(IndQ(K), X)$, x takes over all classical crossings of K, $\mathrm{Ind}(x)$ and $w(x)$ denote the index and writhe of x respectively.

Theorem (C. 2016)

$\Phi_\phi(K)$ *is a virtual knot invariant.*

Example


K_1


K_2

Consider the indexed quandle $Q = \{0, 1\}$ with operations $a *_i b = a + i \pmod 2$, and an indexed quandle 2-cocycle ϕ defined by $\phi(0,0) = \phi(1,1) = 0$ and $\phi(0,1) = \phi(1,0) = 1$. Then we have

$$|Col_{(Q,*_i)}(K_1)| = |Col_{(Q,*_i)}(K_2)| = 2,$$
$$\text{but } \Phi_\phi(K_1) = 1 + 1 \neq \Phi_\phi(K_2) = 0 + 0.$$

What is a chord index?

Definition

In general, a chord index is a(n) integer/polynomial/group/algebra etc. assigned to each real crossing point such that

1. all crossing points involving in the first Reidemeister move have the same index,
2. the two crossing points involving in the second Reidemeister move have the same index,
3. the index of each crossing point involving in the third Reidemeister move is preserved under the third Reidemeister move.

This motivates us to relate it with the Boltzmann weight in (bi)quandle cocycle invariants!

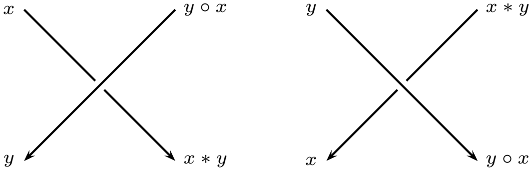
Definition (Fenn, Jordan-Santana, Kauffman 2004)

A biquandle BQ is a set with two binary operations $*, \circ : BQ \times BQ \to BQ$ such that the following axioms are satisfied

1. $\forall x \in BQ, x * x = x \circ x$,
2. $\forall x, y \in BQ$, there are unique $z, w \in BQ$ such that $z * x = y$ and $w \circ x = y$, and the map $S : (x, y) \to (y \circ x, x * y)$ is invertible,
3. $\forall x, y, z \in BQ$, we have

$$(z \circ y) \circ (x * y) = (z \circ x) \circ (y \circ x),$$
$$(y \circ x) * (z \circ x) = (y * z) \circ (x * z),$$
$$(x * y) * (z \circ y) = (x * z) * (y * z).$$

Coloring a virtual knot K with a given finite biquandle BQ: associate each semiarc with an element of BQ such that at each crossing point the following coloring rules are satisfied



Theorem

The coloring number $|Col_{BQ}(K)|$ is a virtual knot invariant.

For a given finite biquandle BQ, consider the following group

$$G_{BQ} = \langle (x, y) \in BQ \times BQ | (x, x) = 1, (x, y)(y, z)(x * y, z \circ y) = (x * z, y * z)(y \circ x, z \circ x)(x, z) \rangle.$$

Remark

For any abelian group A, each homomorphism $\rho : G_{BQ} \to A$ is a biquandle 2-cocycle.

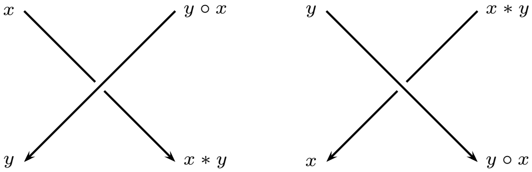
Consider the following subgroup of G_{BQ}

$$\mathfrak{G}_{BQ} = \langle (x, y) \in BQ \times BQ | (x, x) = 1, (x, y) = (x * z, y * z), (y, z) = (y \circ x, z \circ x), (x, z) = (x * y, z \circ y) \rangle.$$

Fix a coloring $f \in Col_{BQ}(K)$, assign a Boltzmann weight

$$\mathfrak{W}_f = (x, y) \in \mathfrak{G}_{BQ}$$

to the crossing points below



Definition

The chord index (associated to BQ) of a crossing point is defined to be $\sum_f \mathfrak{W}_f \in \mathbb{Z}\mathfrak{G}_{BQ}$. In particular, if ψ denotes a homomorphism from $\mathfrak{G}_{BQ}$ to a group A, then we obtain a chord index $\psi(\sum_f \mathfrak{W}_f) \in \mathbb{Z}A$.

For any $\mathfrak{g} \in \mathbb{Z}\mathfrak{G}_{BQ}$ we define

$$a_{\mathfrak{g}}(K) = \begin{cases} \sum\limits_{\mathrm{Ind}(c)=\mathfrak{g}} w(c) & \text{if } \mathfrak{g} \neq \sum 1; \\ \sum\limits_{\mathrm{Ind}(c)=\mathfrak{g}} w(c) - w(K) & \text{if } \mathfrak{g} = \sum 1. \end{cases}$$

Theorem (C. 2016)

For any finite biquandle BQ, $a_{\mathfrak{g}}(K)$ is a virtual knot invariant.

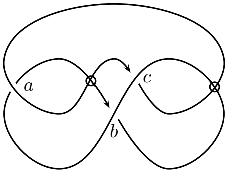
Example

Let $X = (\mathbb{Z}, *, \circ)$ be a biquandle with operations $x * y = x \circ y = x + 1$. Choose a map $\rho : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $\rho(x, y) = y - x$. Then we recover the chord index used in the writhe polynomial.

Example

$X = \{1, 2\}$, and $1 * i = 1 \circ i = 2$ and $2 * i = 2 \circ i = 1$ ($i = 1, 2$). In this case $\mathfrak{G}_X \cong \mathbb{Z}$, which is generated by $t = (1, 2)$. For the virtual *link* below we have

$$a_{1+1+t+t}(L) = 2, \; a_{t+t+t+t}(L) = 1.$$



The following result explains the reason why we must use a biquandle rather than a quandle.

Proposition

Let Q be a finite quandle and K a virtual knot diagram, then all the crossing points of K have the same index $\sum_{|Col_Q(K)|} 1$.

Question

Can we use this generalized chord index to define a nontrivial chord index for classical knots?

Thank you!