Universal parity of free 2-knots

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2-knots

Definition

A 2-knot is a smooth embedding $S^2 \hookrightarrow \mathbb{R}^4$ considered up to isotopy.

Example

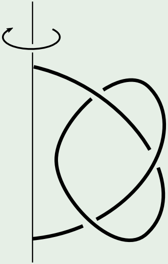


Figure: Spun trefoil

Diagrams of 2-knots

Definition

A Diagram D of a 2-knot is a projection in general position of the 2-knot

$$\begin{array}{ccc} S^2 & \hookrightarrow & \mathbb{R}^4 \\ & {\scriptstyle D}\searrow & \downarrow \\ & & \mathbb{R}^3 \end{array}$$

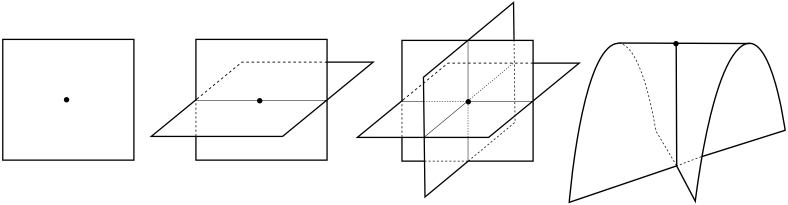


Figure: Types of diagram points

Orientation of double lines

Double lines of 2-knot diagrams have under/overcrossing structure and oriented.

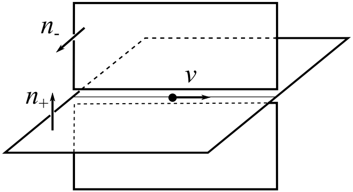


Figure: Orientation of double lines: the triple $(\vec{n}_+, \vec{n}_-, \vec{v})$ have positive orientation in $\mathbb{R}^3$

Roseman moves

Theorem (Roseman)

Diagrams D_1 and D_2 correspond to isotopic 2-knots iff they can be connected by a finite sequence of moves $\mathcal{R}_1$–$\mathcal{R}_7$

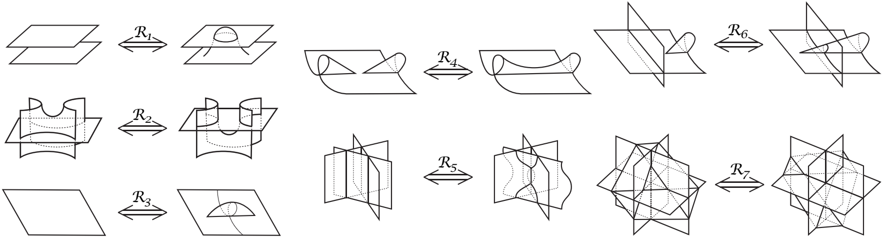


Figure: Roseman moves

Gauss diagrams

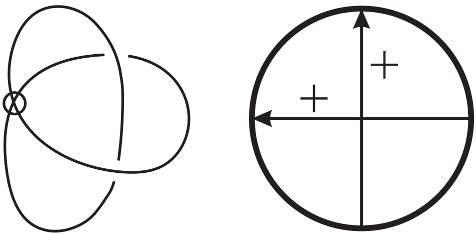


Figure: Virtual trefoil and its Gauss diagram

Remark

- Gauss diagram shows paired pre-images of double points with orientation indicated.
- Gauss diagrams encode virtual knots rather than classical knots.

Gauss diagrams of 2-knots

Definition

A **Gauss diagram** is a set D of marked oriented curves in the sphere S^2 obeying the following conditions.

1. Each curve is closed or terminates in a cusp.
2. Each curve in D is paired with a unique second curve in D, and one curve in the pair is marked as *over* (+), the other is marked as *under* (-).
3. Two curves that terminate in a single cusp are paired together, with the orientations both pointing toward or both pointing away from the cusp.
4. If two curves cross, then the curves they are paired with cross two curves which are paired (that is, triple points appear three times on the sphere).
5. Every triple point is geometrical.

Geometrical triple point

Geometrical conditions of a triple point:

- The marks of the crossing curves in the leaves are $\{+,+\}$, $\{+,-\}$, $\{-,-\}$.
- The orientations of the pairs (c_+, b_+), (c_-, a_+) and (b_-, a_-) coincide.

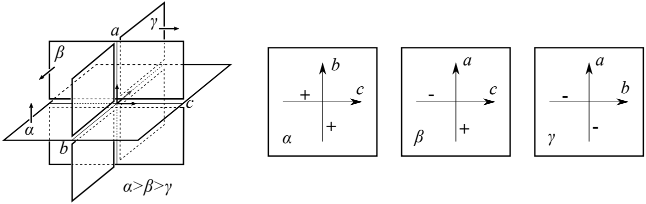


Figure: Orientations at a geometrical triple point

Roseman moves on Gauss diagrams

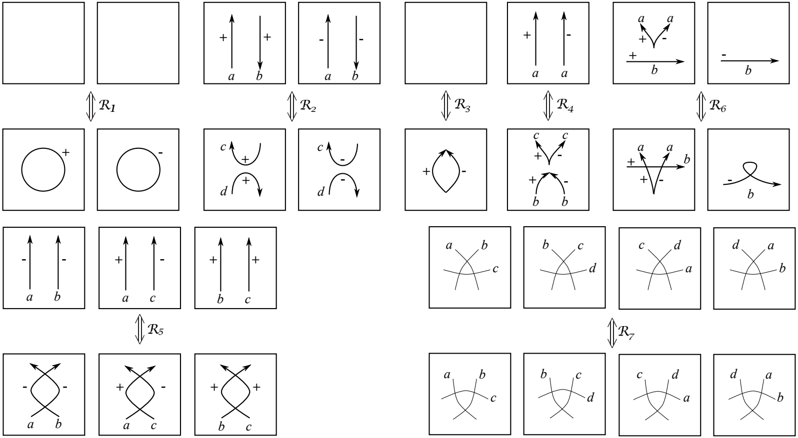
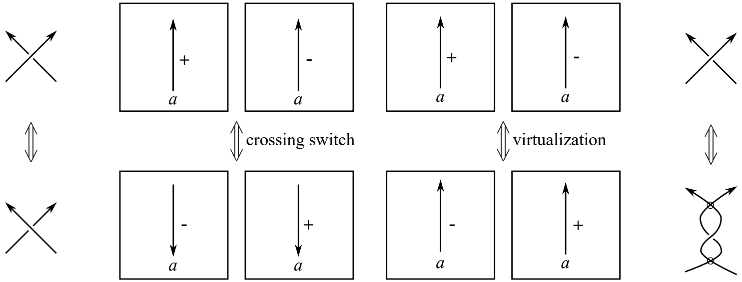


Figure: Roseman moves on Gauss diagrams

Crossing switch and virtualization

Definition

Virtual 2-knots are equivalence classes of Gauss diagrams on the sphere modulo Roseman moves.
Flat 2-knots are virtual 2-knots modulo crossing switches.
Free 2-knots are virtual 2-knots modulo crossing switches and virtualizations.



Parity (V. Manturov, 2009).

Parity is a rule to assign numbers 0 and 1 to the (classical) crossings of diagrams of a knot in a way compatible with Reidemeister moves.

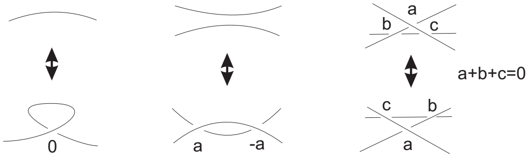


Figure: Parity axioms.

Applications of parity:

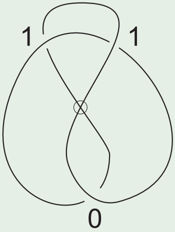
- strengthening of knot invariants
- extention of knot invariants (via functorial maps between knot theories)

Gaussian parity

Definition

Gaussian parity of a crossing is the parity of the number of (classical) crossings that lie on a half of the knot corresponding to the crossing.

Example

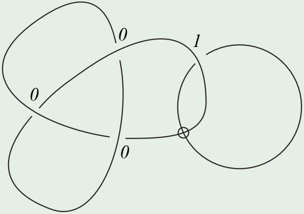


Link parity

Definition

Link parity of a crossing is equal to 0 if the both arcs belong to one component (of a two-component link), and equal to 1 if the arcs belong to different components.

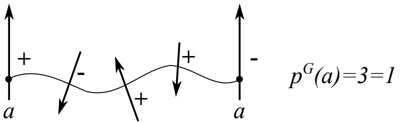
Example



Gaussian and link parity for 2-knots

Definition

Gaussian parity p^G of a double line of a 2-knot is the parity of the number of intersections with double lines of a path connecting two preimages of a point on the given double line.



Definition

Link parity of a double line of a two-component 2-link is even if the intersecting leaves belong to one component, and odd if the intersecting leaves belong to different component.

Definition of parity for 2-knots

Definition

Parity with coefficients in an abelian group A on the diagrams of a 2-knot $\mathcal{K}$ is a family of maps $p_D\colon \mathcal{D}(D) \to A$, where $\mathcal{D}(D)$ is the set of double lines of a diagram D, which obeys the following properties.

1. (*local correspondence property*) let diagrams D and D' be connected by a Roseman move. If double lines $a \in \mathcal{D}(D)$ and $a' \in \mathcal{D}(D')$ have arcs which do not affected by the move and correspond to each other then $p_D(a) = p_{D'}(a')$.
2. (*triple point property*) if double lines $a, b, c \in \mathcal{D}(D)$ intersects in a triple point of the diagram D then $p_D(a) + p_D(b) + p_D(c) = 0$.
3. (*pinch property*) the parity the double line a of any pinch is zero: $p_D(a) = 0$.

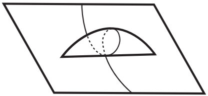
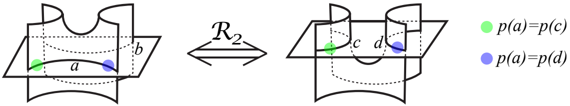


Figure: A pinch

Parity and Roseman moves

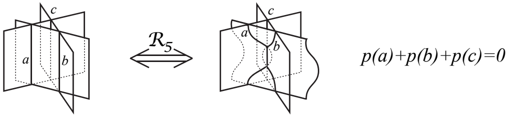
Proposition

The double line involved in a $\mathcal{R}_2$ move have the same parities.

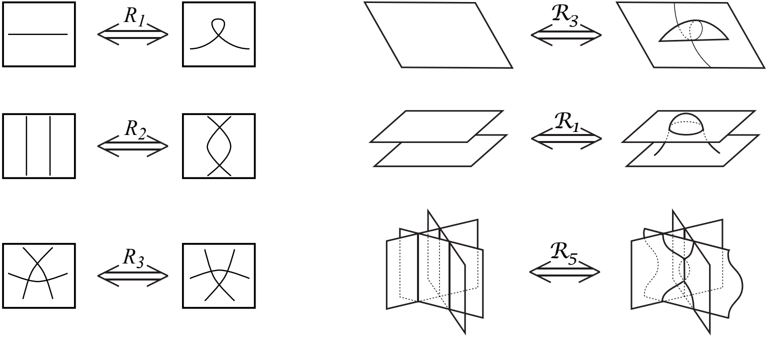


Proposition

The sum of parities of double lines which form a trigonal prism is equal to 0.



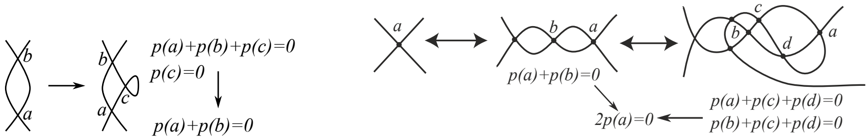
Reduction to 1-dimensional case



Reduction to 1-dimensional case

Proposition

- *If double lines a and b form a bigonal prism then* $p(a) + p(b) = 0$.
- *For any parity p and any double line* $a \in \mathcal{D}(D)$ *we have* $2p_D(a) = 0$.



Remark

The result does not depend on the crossings of the bigonal prism.

Universal parity

Definition

Parity p_u with coefficients in A_u is called a universal parity if for any parity p with coefficients in A there exists a unique homomorphism of group $\rho : A_u \to A$ such that $p_D = \rho \circ (p_u)_D$ for any diagram D.

Theorem (Ilyutko, Manturov, N., 2011)

Gaussian parity (with coefficients in $A_u = \mathbb{Z}_2$) is the universal parity of free knots.

Remark

The result holds for 2-knots too.

Parity and potential

Definition

Let p be a parity with coefficients in A and x and y be points in a diagram D which do not lie on a double line. The potential between points x and y is the parity of the double line created by Rosman move $\mathcal{R}_1$ as shown in the picture below: $\delta_{x,y} = p(a) \in A$.
Analogously, elements $\delta_{\bar{x},y}, \delta_{\bar{y},\bar{x}} \ldots$ can be defined.
Let $\delta_x = \delta_{x,\bar{x}}$.

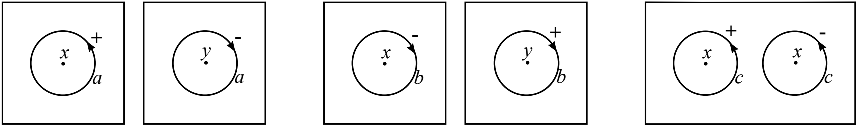


Figure: Potentials $\delta_{x,y} = p(a)$, $\delta_{\bar{y},x} = p(b)$ and $\delta_x = \delta_{x,\bar{x}} = p(c)$

Additivity of potential

Proposition

- *For any x and y* $\delta_{x,y} = \delta_{y,x}$, $\delta_{x,\bar{y}} = \delta_{\bar{y},x}$ *and* $\delta_{\bar{x},\bar{y}} = \delta_{\bar{y},\bar{x}}$.
- *For any* x, y, z $\delta_{x,z} = \delta_{x,y} + \delta_{y,z}$.
- *If parity p is invariant under virtualization then for any x and y* $\delta_{x,\bar{y}} = \delta_{y,\bar{x}}$, $\delta_{x,y} = \delta_{\bar{y},\bar{x}}$ *and* $\delta_x = \delta_y = \delta$.

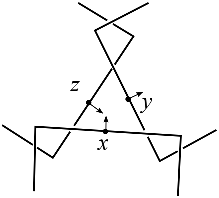


Figure: $\delta_{x,y} + \delta_{y,z} + \delta_{x,z} = 0$

Parity and potential

Proposition

Let x, y, z, u be points near a double line a as shown below. Then $p(a) = \delta_{x,y} = \delta_{y,\bar{z}} = \delta_{\bar{z},\bar{u}} = \delta_{\bar{u},x}$.

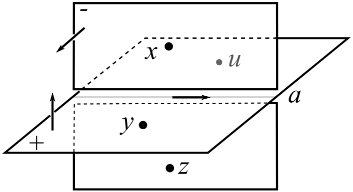


Figure: Parity and potential

Corollary

$\delta_{x,z} = \delta_{y,u} = \delta$.

Main theorem

Proposition

For any parity p with coefficients A on diagrams of free 2-knots (or virtual 2-knots up to virtualization) there exists an element $\delta \in A$, such that for any double line $p(a) = p^G(a) \cdot \delta$.

Corollary

Gaussian parity p^G is the universal parity for free 2-knots.

Affine parity

Definition

Affine parity with coefficients in an abelian group A on the diagrams of a 2-knot $\mathcal{K}$ is a family of maps $p_D \colon \mathcal{D}(D) \to A$, where $\mathcal{D}(D)$ is the set of double lines of a diagram D, which obeys the following properties.

1. (*local correspondence property*) let diagrams D and D' be connected by a Roseman move. If double lines $a \in \mathcal{D}(D)$ and $a' \in \mathcal{D}(D')$ have arcs which do not affected by the move and correspond to each other then $p_D(a) = p_{D'}(a')$.
2. (*triple point property*) if double lines $a, b, c \in \mathcal{D}(D)$ intersects in a triple point of the diagram D then $p_D(a) + p_D(b) + p_D(c) = 0$.

Example (Constant affine parity)

Let $\lambda \in A$ be an element of order 3: $3\lambda = 0$. Then the family of maps $p^\lambda_D(a) \equiv \lambda$ is a parity called constant affine parity.

Affine parity and pinches

Proposition

Let p be an affine parity. Then there exists an element $\lambda \in A$ be an element of order 3 such that $p(a) = \lambda$ for any double line a of a pinch.

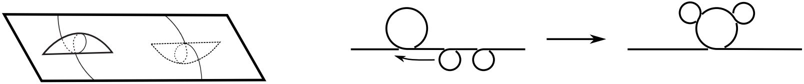


Figure: Pinch and antipinch. One pinch is worth minus two antipinches (and vice versa)

Corollary

Any affine parity is a sum of an ordinary parity and a constant affine parity

Oriented parity

Example

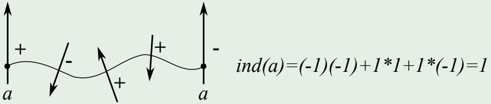


Figure: Double point index as oriented Gaussian parity

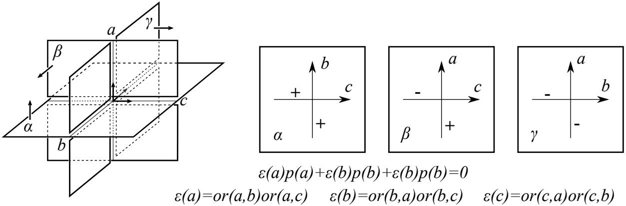


Figure: Oriented triple point property

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