Invariants of virtual doodles

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joint work with
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The 4th Russian-Chinese Knot Theory Conference Registration
Bauman Moscow State Technical University, Moscow, Russia
July 3, 2017 (Monday 3 July - Friday 7 July 2017)

Virtual doodles – doodles on surfaces

- Doodles on surfaces
  A. Bartholomew, R. Fenn, S. Kamada, NK (BFKK) generalized doodles on $\mathbb{R}^2$ to those on surfaces.
  (Doodles on surfaces := Immersed curves on surfaces modulo R I, R II and stabilization)

  A. Bartholomew, R. Fenn, N.K., S. Kamada, *Doodles on surfaces I: An introduction to their basic properties*, ArXiv:1612.08473v1

- Virtual Doodles
  BFKK defined virtual doodles on $\mathbb{R}^2$ and prove that there is a natural bijection between virtual doodles and stable equivalence classes of doodles on surfaces.

Theorem (BFKK)

{virtual doodles} $\Longleftrightarrow$ {doodles on surfaces}/stable equiv.

Virtual doodle (A. Bartholomew, R. Fenn, S. Kamada, NK)

A virtual diagram is a collection of generically immersed circles in $\mathbb{R}^2$ possibly with virtual crossings.

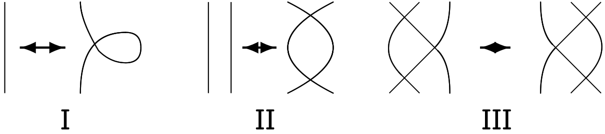
{virtual doodles}
:= {virtual diagrams} /FR I, FR II, and FV moves

{flat virtual links}
:= {virtual diagrams}/FR I, FR II, FR III and FV moves

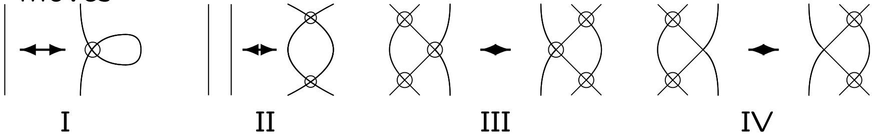
There is a natural projection : {virtual doodles} $\to$ {flat virtual links}.

FR and FV moves

FR moves

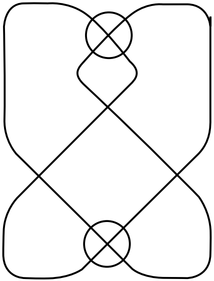


FV moves



Example

Carter's curve on a surface corresponds to the virtual diagram depicted below. This is the smallest nontrivial virtual doodle, $d_{3,1}$ in [BFKK]. (It is also nontrivial as a flat virtual knot.)



Semiquandle (Henrich, Nelson)

A semiquandle X is a set X with two binary operations $(x, y) \mapsto x^y$ and $(x, y) \mapsto x_y$ such that, for all $x, y, z \in X$

(0) there are unique w and $u \in X$ with $x = w^y$ and $x = u_y$

(i) $x_y = y \iff y^x = x$

(ii) $(x_y)^{y^x} = x$ and $(x^y)_{y_x} = x$

(iii) $(x^y)^z = (x^{z_y})^{y^z}$, $(y_x)^{z_{x^y}} = (y^z)_{x^{z_y}}$ and $(z_{x^y})_{y_x} = (z_y)_x$

Allison Henrich and Sam Nelson, *Semiquandles and flat virtual knots*, Pac. J Math. 248 (2010), 155-170

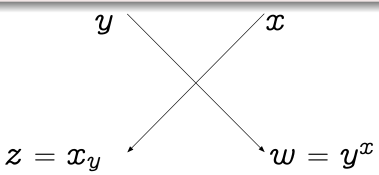
Fundamental semiquandle (Henrich, Nelson)

D: a virtual diagram

The fundamental semiquandle

The fundamental semiquandle $FSQ(D)$ is the semiquandle generated by letters corresponding to semiarcs of D and the relations come from crossings.

Let 4 generators x, y, z, w correspond to the 4 semiarcs around a crossing. The relations are $z = x_y$ and $w = y^x$.



Invariant of flat virtual knots

Remark (Henrich, Nelson).
$D \sim D'$ as a flat virtual link $\Longrightarrow FSQ(D) \cong FSQ(D')$
$D \sim D'$ as a virtual doodle $\Longrightarrow FSQ(D) \cong FSQ(D')$

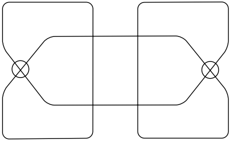
Definition
T : a finite semiquandle
$\mathrm{sc}(D, T) := |\mathrm{Hom}(FSQ(D), T)|$

Theorem ([Henrich, Nelson])

$\mathrm{sc}(D, T)$ is an invariant of flat virtual links.
Hence $\mathrm{sc}(D, T)$ is an invariant of virtual doodles.

Example

Example 1 (c.f. A. Henrich and S. Nelson)



The fundamental semiquandle of Kishino's flat knot K is

$$FSQ(K) = \left\langle a, b, c, d, e, f, g, h \;\middle|\; \begin{array}{l} a^c = b, c_a = d, b^d = c, \\ d_b = e, e^g = f, g_e = h, \\ f^h = g, h_f = a \end{array} \right\rangle.$$

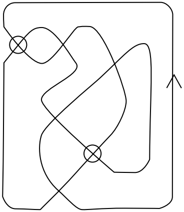
Let T be the semiquandle defined by

$$(U|L) = \left(\begin{array}{cccc|cccc} 1 & 4 & 2 & 3 & 1 & 3 & 4 & 2 \\ 2 & 3 & 1 & 4 & 3 & 1 & 2 & 4 \\ 4 & 1 & 3 & 2 & 2 & 4 & 3 & 1 \\ 3 & 2 & 4 & 1 & 4 & 2 & 1 & 3 \end{array}\right),$$

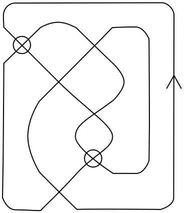
where $U_{i,j} = k$ and $L_{i,j} = k$ such that $x_i^{x_j} = x_k$ and $x_{i_{x_j}} = x_k$

$\mathrm{sc}(K, T) = 16$ and $\mathrm{sc}(UK, T) = 4$ for a trivial knot UK.

Example 2



$d_{4,1}$

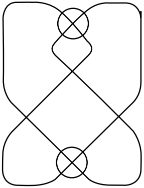


$d_{4,4}$

$\mathrm{sc}(d_{4,1}, T) = \mathrm{sc}(d_{4,4}, T) = 2$

In fact, $d_{4,1}$ and $d_{4,4}$ are equivalent as flat virtual knot. However they are not equivalent as a doodle. We will see this later by using our new invariant.

Example 3



d

$\mathrm{sc}(d_{3,1}, T) = 2$

Thus $d_{3.1}$ is not equivalent to the trivial flat virtual knot. Therefore $d_{3.1}$ is not equivalent to the trivial doodle.

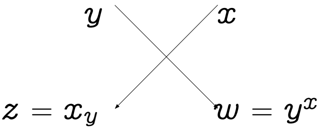
Pre-switch

A pre-switch is a set X with two binary operations $(x, y) \mapsto x^y$ and $(x, y) \mapsto x_y$ such that, for all $x, y \in X$,

(0) there are unique u and $v \in X$ with $x = u^y$ and $x = v_y$;

(i) $x_y = y$ if and only if $y^x = x$;

(ii) $(x_y)^{(y^x)} = x$ and $(x^y)_{(y_x)} = x$.



D : a virtual diagram

The fundamental pre-switch FPS(D)

The pre-switch generated by letters corresponding to the semiarcs of D and the relations come from each flat crossing.

Remark .
$D \sim D'$ as a virtual doodle $\Longrightarrow FPS(D) \cong FPS(D')$

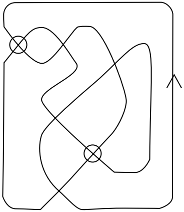
Pre-switch

D : a virtual diagram
T: a finite pre-swich
$\mathrm{pc}(D, T) = |\mathrm{Hom}(\mathrm{FPS}(D), T)|$

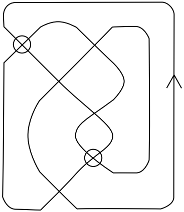
Theorem

Let D and D' be virtual diagrams. If they are equivalent as a virtual doodle, then $\mathrm{FPS}(D)$ is isomorphic to $\mathrm{FPS}(D')$. Consequently, $\mathrm{pc}(D, T)$ is an invariant of virtual doodles.

Example 4



$d_{4,1}$



$d_{4,4}$

Let T' be the pre-switch defined by

$$(U|L) = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 3 & 1 \\ 3 & 3 & 3 & 3 & 1 & 3 \\ 2 & 2 & 1 & 2 & 2 & 2 \end{array}\right).$$

$d_{4,1}$ and $d_{4,4}$ are virtual diagrams in Example 2.
$(d_{4,1}, T') = 2$, $\mathrm{pc}(d_{4,4}, T') = 1$
$d_{4,1}$ is not equivalent to $d_{4,4}$ as a virtual doodle.

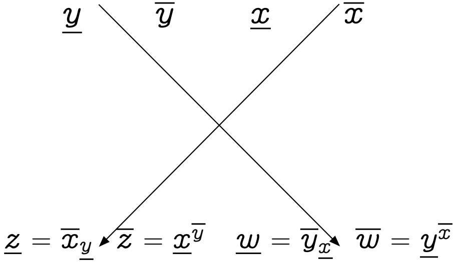
Invariant of virtual doodles

D: a virtual diagram

The doubled fundamental semiquandle

The doubled fundamental semiquandle $\widetilde{FSQ}(D)$ of D is the semiquandle generated by letters $\overline{x}$ and $\underline{x}$ for semiarcs x of D and the defining relations are shown below.

Let $\overline{x}, \overline{y}, \overline{z}, \overline{w}, \underline{x}, \underline{y}, \underline{z}, \underline{w}$ be generators corresponding to the 4 semiarcs as in the figure. Then the relations are $\underline{z} = \overline{x}_{\underline{y}}$, $\overline{z} = \underline{x}^{\overline{y}}$, $\underline{w} = \overline{y}_{\underline{x}}$, $\overline{w} = \underline{y}^{\overline{x}}$.



Theorem

$D \sim D'$ as a virtual doodle $\Longrightarrow \widetilde{FSQ}(D) \cong \widetilde{FSQ}(D')$

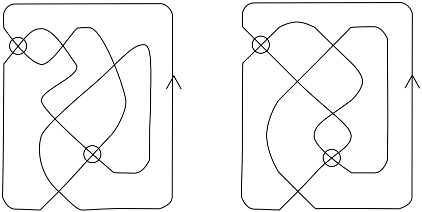
D: a virtual diagram T:a finite semiquandle

$\widetilde{\mathrm{sc}}(D, T) := |\mathrm{Hom}(\widetilde{FSQ}(D), T)|$

Theorem

$\widetilde{\mathrm{sc}}(D, T)$ is an invariant of virtual doodles

Example 5



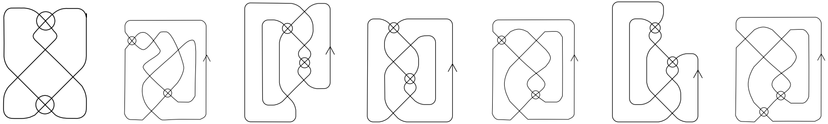
$d_{4,1}$ $d_{4,4}$

Let T be the semiquandle defined by

$$(U|L) = \left(\begin{array}{cccc|cccc} 1 & 4 & 2 & 3 & 1 & 3 & 4 & 2 \\ 2 & 3 & 1 & 4 & 3 & 1 & 2 & 4 \\ 4 & 1 & 3 & 2 & 2 & 4 & 3 & 1 \\ 3 & 2 & 4 & 1 & 4 & 2 & 1 & 3 \end{array}\right).$$

$d_{4,1}$ and $d_{4,4}$ are virtual diagrams in Example 2.
$\widetilde{sc}(d_{4,1}, T) = 16$ and $\widetilde{sc}(d_{4,4}, T) = 256$.
Thus $d_{4,1}$ and $d_{4,4}$ are not equivalent as doodles.

Example 6



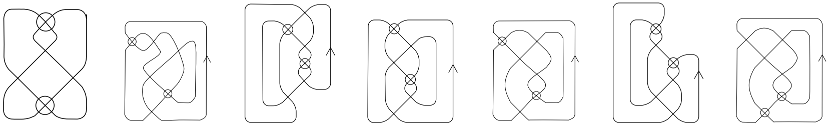
$d_{3.1}$ $d_{4,1}$ $d_{4,2}$ $d_{4,3}$ $d_{4,4}$ $d_{4,5}$ $d_{4,6}$

Let T' be the pre-switch defined by

$$(U|L) = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 3 & 1 \\ 3 & 3 & 3 & 3 & 1 & 3 \\ 2 & 2 & 1 & 2 & 2 & 2 \end{array}\right).$$

$\mathrm{pc}(UD, T') = 3$, where UD is a trivial doodle,
$\mathrm{pc}(d_{3,1}, T') = \mathrm{pc}(d_{4,3}, T') = \mathrm{pc}(d_{4,4}, T') = \mathrm{pc}(d_{4,5}, T') = 1$,
$\mathrm{pc}(d_{4,1}, T') = \mathrm{pc}(d_{4,2}, T') = \mathrm{pc}(d_{4,6}, T') = 2$
$d_{3,1}$, $d_{4,1}$, $d_{4,2}$, $d_{4,3}$, $d_{4,4}$, $d_{4,5}$, and $d_{4,6}$ are not equivalent to a trivial doodle. $d_{3,1}$ (or, $d_{4,3}$,$d_{4,4}$,$d_{4,5}$) is not equivalent to $d_{4,1}$ (or $d_{4,2}$, $d_{4,6}$).

Example 6



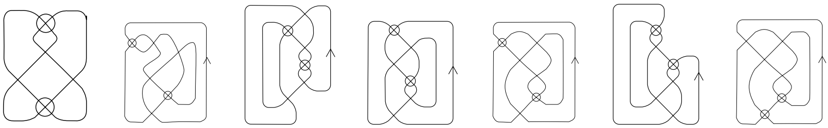
$d_{3.1}$ $d_{4,1}$ $d_{4,2}$ $d_{4,3}$ $d_{4,4}$ $d_{4,5}$ $d_{4,6}$

Let T'' be the pre-switch defined by

$$(U|L) = \left(\begin{array}{ccc|ccc} 2 & 3 & 3 & 2 & 2 & 2 \\ 1 & 1 & 1 & 3 & 1 & 3 \\ 3 & 2 & 2 & 1 & 3 & 1 \end{array}\right).$$

$\mathrm{pc}(UD, T'') = 3$, $\mathrm{pc}(d_{3,1}, T'') = \mathrm{pc}(d_{4,1}, T'') = \mathrm{pc}(d_{4,2}, T'') = \mathrm{pc}(d_{4,3}, T'') = \mathrm{pc}(d_{4,4}, T'') = \mathrm{pc}(d_{4,5}, T'') = 1$, and $\mathrm{pc}(d_{4,6}, T'') = 2$. Hence, $d_{4,6}$ is not equivalent to $d_{4,1}$ and $d_{4,2}$.

Example 6



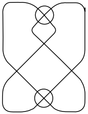
$d_{3.1}$ $d_{4,1}$ $d_{4,2}$ $d_{4,3}$ $d_{4,4}$ $d_{4,5}$ $d_{4,6}$

Let T be the semiquandle defined by

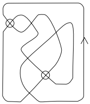
$$(U|L) = \left(\begin{array}{cccc|cccc} 1 & 4 & 2 & 3 & 1 & 3 & 4 & 2 \\ 2 & 3 & 1 & 4 & 3 & 1 & 2 & 4 \\ 4 & 1 & 3 & 2 & 2 & 4 & 3 & 1 \\ 3 & 2 & 4 & 1 & 4 & 2 & 1 & 3 \end{array}\right).$$

$\widetilde{sc}(UD, T) = \widetilde{sc}(d_{3,1}, T) = \widetilde{sc}(d_{4,1}, T) = \widetilde{sc}(d_{4,2}, T) = 16$ and $\widetilde{sc}(d_{4,3}, T) = \widetilde{sc}(d_{4,4}, T) = \widetilde{sc}(d_{4,5}, T) = \widetilde{sc}(d_{4,6}, T) = 256$. Thus $d_{3,1}$ is not equivalent to $d_{4,3}$ (or $d_{4,4}$, $d_{4,5}$).

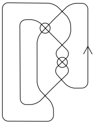
Example 6



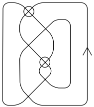
$d_{3.1}$



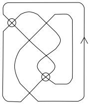
$d_{4,1}$



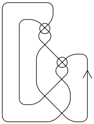
$d_{4,2}$



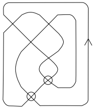
$d_{4,3}$



$d_{4,4}$



$d_{4,5}$



$d_{4,6}$

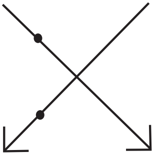
Remark. By our method, $d_{4,1}$ and $d_{4,2}$ (or $d_{4,3}$, $d_{4,4}$ and $d_{4,5}$) can't be distinguished.

- $d_{3,1} \sim d_{3,1}!$. $-d_{3,1}$ is obtained from $d_{3,1}$ by K-flype. (show the definition later.)
- $d_{4,1}$, $-d_{4,1}$, $d_{4,1}!$ $-d_{4,1}!$ are distinct. $d_{4,2} \sim -d_{4,1}!$
- $d_{4,3} \sim -d_{4,3}$
- $d_{4,4}$, $-d_{4,4}$, $d_{4,4}!$ $-d_{4,4}!$ are distinct. $d_{4,5} \sim -d_{4,4}!$
- $d_{4,6}$, $-d_{4,6}$, $d_{4,6}!$ $-d_{4,6}!$ are distinct.

Cut points and Cut system (c.f. H. Dye)

D: a virtual diagram
P: a set of points on D

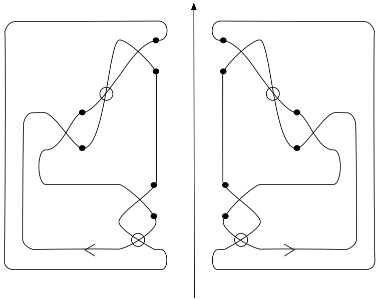
We call P a D cut system of D if two points are given at each crossing as in the figure.



A point of D cut system is called D cut point.

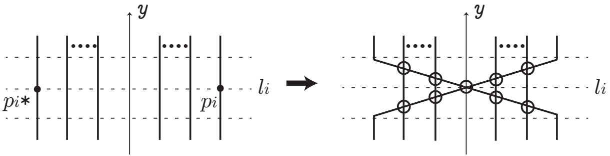
Double covering diagram of a virtual diagram

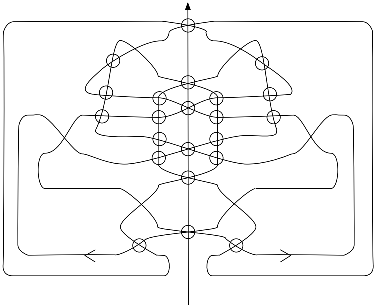
(D, P): a virtual diagram with D cut system
(D^*, P^*): the virtual diagram with a cut system obtained from (D, P) by the reflection with respect to y-axis



D^* D

Double covering diagram of a virtual diagram





The diagram obtained this way is called the double covering of D and denoted by $\widetilde{D}$.

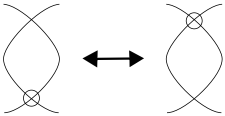
Double covering diagram of a virtual diagram

Theorem

$D \sim D'$ as a virtual doodle $\Longrightarrow \widetilde{D} \sim \widetilde{D}'$ as a virtual doodle.

Remark. (NK)

$D \sim D'$ as a flat virtual link $\Longrightarrow \widetilde{D} \sim \widetilde{D}'$ (K-equivalent)



D_1, D_2 : two virtual diagram
D_1 and D_2 are K-equivalent if D_2 is obtained from D_1 by FR I, FR II, FR III, V-moves and Kauffman flypes (K-flypes).

Theorem

$\widetilde{FSQ}(D) \cong FSQ(\widetilde{D})$

Thank you for your attention.