On classifying link maps in the 4-sphere

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Outline

1. Link Homotopy

2. Intersections of surfaces in a 4-manifold

3. Kirk’s $\sigma$ invariant of link homotopy

4. Techniques to address the open problem:
   \[ \text{does } \sigma = 0 \Rightarrow \text{nullhomotopic?} \]
The Classification Problem

Link map:

\[ f : S^{p_1} \cup S^{p_2} \cup \ldots \cup S^{p_n} \to S^m, \quad f(S^{p_i}) \cap f(S^{p_j}) = \emptyset \]
for \( i \neq j \)
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**Link homotopy** = homotopy through link maps
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Link homotopy = homotopy through link maps

Problem: (For fixed \( p_i, n, m \)) Classify the set
\[
\{ \text{link maps } f : S^{p_1} \cup S^{p_2} \cup \ldots \cup S^{p_n} \to S^m \} \]
What do we know?

\[ S^1 \cup S^1 \cup \ldots \cup S^1 \rightarrow S^3 \]
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  classified up to link homotopy

- Koschorke, a.o. (early 90s):
  
  \[ S^{p_1} \cup S^{p_2} \cup \ldots \cup S^{p_n} \rightarrow S^m, \; 2 < p_i < m - 1 \]
  
  classification \longleftrightarrow \text{homotopy theory questions}
  
  in certain dimension ranges
Hard: links maps in $S^4$

\[ f : S^2_+ \cup S^2_- \to S^4, \quad f(S^2_+) \cap f(S^2_-) = \emptyset \]

Write \( f_+ = f|_{S^2_+} \), \( f_- = f|_{S^2_-} \)
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Example:

\[ 
\begin{array}{cccc}
\text{original} & \text{rotated} & \text{link} & \text{result} \\
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Write $f_+ = f \mid_{S_+^2}, \quad f_- = f \mid_{S_-^2}$

Example:
Classifying link maps

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Write \( f_+ = f|_{S_+^2}, \quad f_- = f|_{S_-^2} \)

Q: When is a link map link homotopic to the trivial link?

(Trivial link map: two embedded 2-spheres bounding disjoint 3-balls)
Classifying link maps

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Q: When is a link map link homotopic to the trivial link? an embedding? (Bartels-Teichner ’99)

(Trivial link map: two embedded 2-spheres bounding disjoint 3-balls)
Self-intersections of a 2-sphere

Consider a simple map \( f : \mathbb{S}^2 \rightarrow \mathbb{R}^4 \)

\( \Rightarrow \) ... that is immersed
with two double points
Self-intersections of a 2-sphere

Consider a simple map $f : S^2 \to \mathbb{R}^4$

▷ ... that is immersed with two double points of opposite sign
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Local picture of two dbl points of \( f : S^2 \to X^4 \) with opp signs.
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\[
\ln \pi_1(X, \bullet): \alpha \beta \gamma^{-1} \simeq 1 \Rightarrow \beta \simeq \alpha^{-1} \gamma
\]
Self-intersections of a 2-sphere

Local picture of two dbl points of $f : S^2 \to X^4$ with opp signs.

In $\pi_1(X, \bullet)$: $\alpha \beta \gamma^{-1} \simeq 1 \Rightarrow \beta \simeq \alpha^{-1} \gamma$

So: “dbl point loops” homotopic $\Rightarrow$ “Whitney” disk

$W \subset X$

$f(S^2) \subset X$
Self-intersections of a 2-sphere

Local picture of two dbl points of $f : S^2 \to X^4$ with opp signs.

\[ \text{In } \pi_1(X, \bullet): \alpha \beta \gamma^{-1} \simeq 1 \Rightarrow \beta \simeq \alpha^{-1} \gamma \]

So: “dbl point loops” homotopic \( (\alpha \simeq \gamma) \) \Rightarrow \text{get (immersed) Whitney disk } W \subset X

\[ f(S^2) \subset X \]
Self-intersections of a 2-sphere

Local picture of two dbl points of $f : S^2 \to X^4$ with opp signs.

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$S^2 \ni f(S^2) \subset X$
Self-intersections of a 2-sphere

Local picture of two dbl points of \( f : S^2 \to X^4 \) with opp signs.

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\( S^2 \)

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Self-intersections of a 2-sphere

Local picture of two dbl points of $f : S^2 \to X^4$ with opp signs.

$W$ embedded and misses $f(S^2) \Rightarrow$ can homotope $f$ to remove double points

$S^2 \quad f(S^2) \subset X$
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Wall self-intersection number $\mu$

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$S^2 \ni \alpha_f

f(S^2) \subset X$
Wall self-intersection number $\mu$

$$f : S^2 \to X^4$$

$$\mu(f) = \sum_{p \in \text{self}(f)} \text{sign}_p \alpha_p \in \mathbb{Z}[\pi_1(X)]$$
Wall self-intersection number $\mu$

$$f : S^2 \rightarrow X^4, \, \pi_1(X) \cong \mathbb{Z} = \langle t^n : n \in \mathbb{Z} \rangle$$

$$\mu(f) = \sum_{p \in \text{self}(f)} \text{sign}_p t^{np} \in \mathbb{Z}[t, t^{-1}]$$

$$S^2 \xrightarrow{f} f(S^2) \subset X$$
Wall self-intersection number $\mu$

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Wall intersection form $\lambda$

$A, B$ - 2-disks or 2-spheres in $X^4$, $\pi_1(X) \cong \mathbb{Z}$
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$A$, $B$ - 2-disk or 2-sphere in $X^4$, $\pi_1(X) \cong \mathbb{Z}$

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$S^2$, $D^2$
Wall intersection form $\lambda$

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$S^2, D^2$
Kirk’s link homotopy invariant $\sigma$

$$f : S_+^2 \cup S_-^2 \to S^4, \quad f_\pm : S_+^2 \to S^4 \setminus f(S_+^2)$$
Kirk’s link homotopy invariant $\sigma$

$$f : S^2_+ \cup S^2_- \to S^4,$$  \quad $$f_\pm : S^2_+ \to S^4 \setminus f(S^2_+)$$

After a link homotopy, $\pi_1(S^4 \setminus f(S^2_+)) \cong \mathbb{Z}$
Kirk’s link homotopy invariant $\sigma$

$$f : S^2_+ \cup S^2_- \to S^4, \quad f_\pm : S^2_+ \to S^4 \setminus f(S^2_\mp)$$

After a link homotopy, $\pi_1(S^4 \setminus f(S^2_\mp)) \cong \mathbb{Z}$

$$\sigma_\pm(f) = \mu(f_\pm) = \sum_{p \in \text{self}(f_\pm)} \text{sign}_p(t^{n_p} - 1) \in \mathbb{Z}[t]$$
Kirk’s link homotopy invariant \( \sigma = (\sigma_+, \sigma_-) \)

\[
\sigma_{\pm}(f) = \sum_{p \in \text{self}(f_{\pm})} \text{sign}_p \left( t^{n_p} - 1 \right) \in \mathbb{Z}[t]
\]

Example:
Kirk’s link homotopy invariant $\sigma = (\sigma_+, \sigma_-)$

$$\sigma_{\pm}(f) = \sum_{p \in \text{self}(f_{\pm})} \text{sign}_p \left( t^{n_p} - 1 \right) \in \mathbb{Z}[t]$$

Example:

$f(S^2_+) f(S^2_-)$

$$\sigma_+(f) = t^1 - 1$$

$$\sigma_-(f) = - t^1 + 1$$
Properties of $\sigma$:

- Link homotopy invariant
- $f$ link homotopic to embedding
  \[ \Rightarrow \sigma_+(f) = 0 = \sigma_-(f) \]
- $\sigma_\pm(f) = 0$
  \[ \Rightarrow \text{can equip } f_\pm \text{ with Whitney disks in } S^4 \setminus f(S^2_\mp) \]
Is $\sigma$ the complete obstruction to embedding?

That is, is the existence of Whitney disks alone enough to embed?

\[ f(S^2_+) \]
\[ f(S^2_-) \]

\[ \sigma_+(f) = -t^2 + 4t - 3 \]
\[ \sigma_-(f) = t^0 - t^0 = 0 \]
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$+t^0$ $-t^0$ $W$

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The Whitney disk intersects $f(S_-^2)$... so can’t use to homotope $f_-$ to an embedding.

$$f(S_+^2)$$

$$f(S_-^2)$$

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Solution: try to form a “secondary” Whitney disk $V$
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Solution: try to form a “secondary” Whitney disk $V$

\[ \rightsquigarrow \text{define a “secondary” invariant that obstructs this} \]
Is $\sigma$ the complete obstruction to embedding?

Some history:

- 1997: Li defined a secondary link htpy invariant $\omega = (\omega_+, \omega_-)$
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  - $\omega_\pm$ supposes $\sigma_\pm = 0$ and counts intersections between $f(S_\pm)$ and WDs in $S^4 - f(S_\pm^2)$
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  - “Example” of link map $f$ with $\sigma(f) = (0, 0)$ but $\omega(f) \neq (0, 0)$
    $\Rightarrow$ **Counterexample**
Is $\sigma$ the complete obstruction to embedding?

Some history:

- 1997: Li defined a secondary link htpy invariant $\omega = (\omega_+, \omega_-)$
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  - $f$ link htpic to embedding $\Rightarrow \omega(f) = (0, 0)$
  - “Example” of link map $f$ with $\sigma(f) = (0, 0)$ but $\omega(f) \neq (0, 0)$ $\Rightarrow$ Counterexample

- 1997: Pilz found mistake in Li’s example (actually had $\omega = (0, 0)$)
Nothing new: \( \sigma(f) = (0, 0) \Rightarrow \omega(f) = (0, 0) \)

**Theorem (L.)**

If \( f : S^2_+ \cup S^2_- \rightarrow S^4 \) is a link map with both \( \sigma_+(f) = 0 \) and \( \sigma_-(f) = 0 \), then:

(after a link homotopy) each component \( f_\pm \) can be equipped with framed, immersed Whitney disks whose interiors are disjoint from both \( f(S^2_+) \) and \( f(S^2_-) \).
Nothing new: \( \sigma(f) = (0, 0) \Rightarrow \omega(f) = (0, 0) \)

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\[ f(S^2_+) \]
\[ f(S^2_-) \]
Nothing new: $\sigma(f) = (0, 0) \Rightarrow \omega(f) = (0, 0)$

**Theorem (L.)**

If $f : S^2_+ \cup S^2_- \to S^4$ is a link map with both $\sigma_+(f) = 0$ and $\sigma_-(f) = 0$, then:

(after a link homotopy) each component $f_{\pm}$ can be equipped with framed, immersed Whitney disks whose interiors are disjoint from both $f(S^2_+)$ and $f(S^2_-)$.
Nothing new: \( \sigma(f) = (0, 0) \Rightarrow \omega(f) = (0, 0) \)

**Theorem (L.)**

Let \( f : S^2_+ \cup S^2_- \to S^4 \) be a link map with \( \sigma_-(f) = 0 \).

If \( \sigma_+(f) = \sum_{p \in \text{self}(f_+)} (t^{np} - 1) \),

then \( \omega_-(f) = \# \{ p : n_p \equiv 2 \mod 4 \} \mod 2 \).

In particular, there are infinitely many link maps \( f \) with \( \omega(f) = (0, 0) \) but \( \sigma(f) \neq (0, 0) \).
Towards a better invariant?

Let \( f : S^2_+ \cup S^2_- \to S^4 \) be a link map.

**Proposition (S. Kamada)**

After a link homotopy, \( f(S^2_-) \) is an unknotted immersion in \( S^4 \) with \( d \geq 0 \) pairs of oppositely-signed double points.
Towards a better invariant?

Let \( f : S^2_+ \cup S^2_- \to S^4 \) be a link map. Write \( X_- = S^4 \setminus f(S^2_-) \).

- \( \pi_1(X_-) \cong \mathbb{Z}, \quad \mathbb{Z}\pi_1 = \mathbb{Z}[t, t^{-1}] \)
- \( \pi_2(X_-) \cong \left( \bigoplus_{i=1}^{2d} \mathbb{Z} \right)[t, t^{-1}] \)

---

Diagram:

\[ f(S^2_-) \]

\[ \quad \rightarrow \quad \]

\[ p_1^- \quad p_1^+ \quad \]

\[ \quad \rightarrow \quad \]

\[ p_2^- \quad p_2^+ \quad \]
Towards a better invariant?

Construct generators of \( \pi_2(X_-) = \left( \bigoplus_{i=1}^{2d} \mathbb{Z} \right)[t, t^{-1}] \)

- \( H_2(X_-) = \mathbb{Z}^{2d} \)
- Generated by linking tori \( \{ T_i^+, T_i^- \}_{i=1}^d \)

\[ f(S_2^-) \]
Towards a better invariant?

Construct generators of $\pi_2(X^-) = \left(\bigoplus_{i=1}^{2d} \mathbb{Z}\right)[t, t^{-1}]$.

- Surger $T_p$ to a 2-sphere $A_p$
- $A_p = (T_p \setminus \text{annulus}) \cup (D_p \cup D'_p)$
Towards a better invariant?

Construct generators of $\pi_2(X_-) = (\bigoplus_{i=1}^{2d} \mathbb{Z})[t, t^{-1}]$.

- $A_p = (T_p \setminus \text{annulus}) \cup (D_p \cup D'_p)$
Towards a better invariant?

Construct generators of $\pi_2(X \approx) = (\bigoplus_{i=1}^{2d} \mathbb{Z})[t, t^{-1}]$.

- $A_p = (T_p \setminus \text{annulus}) \cup (D_p \cup D'_p)$
- $\lambda(f_+, A_p) = (1 + t)\lambda(f_+, D_p) \in \mathbb{Z}\pi_1(X \approx) = \mathbb{Z}[t, t^{-1}]$
Towards a better invariant?

Construct generators of $\pi_2(X_-) = (\bigoplus_{i=1}^{2d} \mathbb{Z})[t, t^{-1}]$.

- $A_p = (T_p \setminus \text{annulus}) \cup (D_p \cup D'_p)$
- $\lambda(f_+, A_p) = (1 + t)\lambda(f_+, D_p) \in \mathbb{Z}\pi_1(X_-) = \mathbb{Z}[t, t^{-1}]$
- $\mu(A_p) = \text{sign}_p(t - 1) \in \mathbb{Z}[t]$
Towards a better invariant?

Construct generators of $\pi_2(X_-) = (\bigoplus_{i=1}^{2d} \mathbb{Z})[t, t^{-1}]$.

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Towards a better invariant?

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- $A_p = (T_p \setminus \text{annulus}) \cup (D_p \cup D'_p)$
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Towards a better invariant?

Construct generators of $\pi_2(X_-) = (\bigoplus_{i=1}^{2d} \mathbb{Z})[t, t^{-1}]$.

- $A_p = (T_p \setminus \text{annulus}) \cup (D_p \cup D'_p)$
- $\lambda(f_+, D_p) = (1 + t)\lambda(f_+, E_p)$
- $\lambda(f_+, E_p) \xrightarrow{t \mapsto 1} n_p$ where $\sigma_-(f) = \sum_p \text{sign}_p(t^{np} - 1)$
Towards a better invariant?

Let $f : S^2_+ \cup S^2_- \to S^4$ be a link map with $\sigma_-(f) = \text{sign}_p(t^{np} - 1)$. 
Towards a better invariant?

Let $f : S^2_+ \cup S^2_- \to S^4$ be a link map with $\sigma_-(f) = \text{sign}_p(t^{np} - 1)$.

After a link homotopy...

- $\pi_2(X_-) = (\bigoplus_{i=1}^{2d} \mathbb{Z})[t, t^{-1}]$ has basis rep. by 2-spheres $\{A_p\}_p$
- $A_p \cap A_q = \emptyset$
- $\mu(A_p) = \text{sign}_p(t - 1)$
- $\lambda(f_+, A_p) = (1 + t)^2 c_p(t)$, $c_p(1) = n_p$
Towards a better invariant?

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After a link homotopy...

- $\pi_2(X_-) = \left( \bigoplus_{i=1}^{2d} \mathbb{Z} \right)[t, t^{-1}]$ has basis rep. by 2-spheres $\{A_p\}_p$
- $A_p \cap A_q = \emptyset$
- $\mu(A_p) = \text{sign}_p(t - 1)$
- $\lambda(f_+, A_p) = (1 + t)^2c_p(t), \quad c_p(1) = n_p$
- So: $f_+ \in \pi_2(X_-)$

\[ f_+ = \sum_p c_p(t)A_p, \quad c_p(1) = n_p \]
Towards a better invariant?

Let $f : S^2_+ \cup S^2_- \to S^4$ be a link map with $\sigma_-(f) = 0$.

After a link homotopy...

- $f_+ = \sum_j t^{n_j} A^+_j + t^{m_j} A^-_j$, \hspace{1cm} $\mu(A^\pm_j) = \pm(t - 1)$
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- Represented by tubing pairwise-tubed 2-spheres....

![Diagram of tubing pairwise-tubed 2-spheres]
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A_j^+ + t^2 A_j^- \subset X_-
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\( f(S^2_-) \)
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• **Question:** Does $\sigma$ classify link maps?
Still open

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- **Question:** Can a secondary invariant for 3-component link maps be defined? Is it stronger than $\sigma$?